

Different Statistical Future of Dynamical Orbits

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Abstract

For expanding or hyperbolic dynamical systems, we use upper and lower natural density and Banach density to divide dynamical orbits into several different level sets. Meanwhile, non-recurrence and Birkhoff averages are considered and we obtained simultaneous level sets by mixing them together. By studying the topological entropy via multifractal analysis, we reveal the complexity of each level set. In this process we generalize entropy-dense property and saturated property to strong ones.

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Key words and phrases: Minimality; Non-recurrence; Irregular points or points with historic behavior; ω -limit set.
AMS Review: 37D20; 37C50; 37B20; 37B40; 37C45.

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1 Introduction

1.1 Main Results

By a *topological dynamical system* (*TDS* for short) (X, f) , we mean a continuous map f acting on a compact metric space (X, d) . Throughout this paper, we suppose (X, d) has infinitely many points and denote the sets of natural numbers, integer numbers and nonnegative numbers by $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+$ respectively. For any $x \in X$, the orbit of x is $\{f^n x\}_{n=0}^\infty$ which we denote by $orb(x, f)$. The ω -limit set of x is defined as

$$\omega_f(x) := \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \{f^k x\}} = \{y \in X : \exists n_i \rightarrow \infty \text{ s.t. } f^{n_i} x \rightarrow y\}.$$

It is clear that $\omega_f(x)$ is a nonempty compact f -invariant set.

Definition 1.1. We call $x \in X$ to be *recurrent*, if $x \in \omega_f(x)$. A point $x \in X$ is called *wandering*, if there is a neighborhood U of x such that the sets $f^{-n}U$, $n \geq 0$, are mutually disjoint. Otherwise, x is called *non-wandering*. A point $x \in X$ is *almost periodic*, if for every open neighborhood U of x , there exists $N \in \mathbb{N}$ such that $f^k(x) \in U$ for some $k \in [n, n + N]$ and every $n \in \mathbb{N}$.

Denote the set of almost periodic points by $AP(f)$, the set of non-wandering points by $\Omega(f)$, the set of recurrent points by $Rec(f)$ and the set of non-recurrent points by $NR(f)$.

Now we recall some notions of density to describe different statistical structure.

Definition 1.2. Let $S \subseteq \mathbb{N}$, define

$$\bar{d}(S) := \limsup_{n \rightarrow \infty} \frac{|S \cap \{0, 1, \dots, n-1\}|}{n}, \quad \underline{d}(S) := \liminf_{n \rightarrow \infty} \frac{|S \cap \{0, 1, \dots, n-1\}|}{n},$$

where $|Y|$ denotes the cardinality of the set Y . These two concepts are called *upper density* and *lower density* of S , respectively. If $\bar{d}(S) = \underline{d}(S) = d$, we call S to have density of d . Define

$$B^*(S) := \limsup_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|}, \quad B_*(S) := \liminf_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|},$$

here $I \subseteq \mathbb{N}$ is taken from finite continuous integer intervals. These two concepts are called *Banach upper density* and *Banach lower density* of S , respectively. A set $S \subseteq \mathbb{N}$ is called *syndetic*, if there is $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $S \cap \{n, n+1, \dots, n+N\} \neq \emptyset$.

These concepts of density are basic and have played important roles in the field of dynamical systems, ergodic theory and number theory, etc. Let $U, V \subseteq X$ be two nonempty open subsets and $x \in X$. Define sets of visiting time

$$N(U, V) := \{n \geq 1 \mid U \cap f^{-n}(V) \neq \emptyset\} \quad \text{and} \quad N(x, U) := \{n \geq 1 \mid f^n(x) \in U\}.$$

Definition 1.3. For $x \in X$ and $\xi = \bar{d}, \underline{d}, B^*, B_*$, a point $y \in X$ is called $x - \xi$ -accessible, if for any $\epsilon > 0$, $N(x, V_\epsilon(y))$ has positive density w.r.t. ξ , where $V_\epsilon(x)$ denotes the ball centered at x with radius ϵ . Let

$$X_\xi(x) := \{y \in X \mid y \text{ is } x - \xi - \text{accessible}\}.$$

For convenience, it is called $\xi - \omega$ -limit set of x .

Note that

$$X_{B_*}(x) \subseteq X_{\underline{d}}(x) \subseteq X_{\bar{d}}(x) \subseteq X_{B^*}(x) \subseteq \omega_f(x). \quad (1.1)$$

Moreover, they are all compact and invariant (with possibility that some sets are empty). Denote $QAP(f) = \{x \in X : \omega_f(x) \text{ is minimal}\}$ and call $x \in QAP(f)$ a quasi-minimal point. Moreover, denote $WQAP = \{x \in X : C_x \text{ is minimal}\}$ and call $x \in WQAP(f)$ a weak-quasi-minimal point. It is clear that $QAP(f) \subseteq WQAP(f)$.

For a continuous function φ on X , define the φ -irregular set as

$$I_\varphi(f) := \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ diverges} \right\}.$$

Definition 1.4. A topological dynamical system (X, f) is called *topologically expanding*, if it is positively expansive and satisfies the shadowing property. A homeomorphism (X, f) is called *topologically hyperbolic*, if it is expansive and satisfies the shadowing property.

Let $\Upsilon_T := \{x \in X \mid f|_{\omega_f(x)} \text{ is transitive}\}$ and $\Upsilon_S := \{x \in X \mid f|_{\omega_f(x)} \text{ has the shadowing property}\}$. Note that for any $x \in X$, $\omega_f(x)$ is internally chain transitive. If $x \in \Gamma_S$, then $\omega_f(x)$ is transitive. So one has $\Gamma_S \subseteq \Gamma_T$. Therefore, if we let $\Upsilon := \Upsilon_S \cap \Upsilon_T = \Gamma_S$, X can be written as the disjoint union

$$X = \Upsilon \sqcup \Upsilon_T^c \sqcup (\Upsilon_S^c \cap \Upsilon_T).$$

Theorem A. *Suppose (X, f) is topologically expanding and transitive (resp., (X, f) is a homeomorphism that is topologically hyperbolic and transitive). Let φ be a continuous function on X . If $I_\varphi(f) \neq \emptyset$, then*

- (1) $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (1') $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (2) $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (2') $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (3) $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f);$ and $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (3') $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) = h_{top}(f);$ and $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (4) $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f);$ and $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (4') $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) = h_{top}(f);$ and $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$
- (5) $h_{top}(\{x \in X \mid X_{B^*}(x) = X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$ *Alternatively,*

$$h_{top}(QAP(f) \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$$

- (5') $h_{top}(\{x \in X \mid X_{B^*}(x) = X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$ *Alternatively,*

$$h_{top}((WQAP(f) \setminus QAP(f)) \cap NR(f) \cap I_\varphi(f)) = h_{top}(f).$$

For a continuous function φ on X , denote

$$L_\varphi = \left[\inf_{\mu \in M(f, X)} \int \varphi d\mu, \sup_{\mu \in M(f, X)} \int \varphi d\mu \right] \text{ and } \text{Int } L_\varphi = \left(\inf_{\mu \in M(f, X)} \int \varphi d\mu, \sup_{\mu \in M(f, X)} \int \varphi d\mu \right).$$

For any $a \in L_\varphi$, denote

$$t_a = \sup_{\mu \in M(f, X)} \left\{ h_\mu : \int \varphi d\mu = a \right\}$$

and consider the level set

$$R_\varphi(a) := \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = a \right\}.$$

Theorem B. *Suppose (X, f) is topologically expanding and transitive (resp., (X, f) is a homeomorphism that is topologically hyperbolic and transitive).. For a continuous function φ on X and $a \in \text{Int}(L_\varphi)$, we have the following conditional variational principle:*

- (1) $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (1') $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (2) $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (2') $h_{top}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (3) $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$; and
 $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (3') $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$; and
 $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (4) $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$; and
 $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (4') $h_{top}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$; and
 $h_{top}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) = t_a$.
- (5) For any $k \in \mathbb{N}$,

$$\begin{aligned}
& h_{top}(R_\varphi(a)) = h_{top}(R_\varphi(a) \cap NR(f)) \\
&= h_{top}(R_\varphi(a) \cap NR(f) \cap WQAP(f)) \\
&= h_{top}(R_\varphi(a) \cap NR(f) \cap QAP(f)) \\
&= \sup \left\{ h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X) \right\} \\
&= \sup \left\{ h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X) \text{ and } S_\mu \neq X \right\} \\
&= \sup \left\{ h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X), S_\mu \neq X, S_\mu \text{ is minimal and } \#M(f, S_\mu) = k \right\}.
\end{aligned}$$

1.2 Examples

1.2.1 Symbolic dynamics

Symbol spaces and their shift maps have a long history in dynamics beginning at the latest in 1921 with Morse [48]. The definition of subshift of finite type (f.t. subshift for short) is due to Parry [52, 53] in the study of intrinsic Markov chain. It also evolved independently from Smale [68] via Bowen-Lanford [16]. See [76] for proofs and related results.

1.2.2 Smooth dynamics

We now suppose that $f : M \rightarrow M$ is a diffeomorphism of a compact C^∞ Riemannian manifold M . Then the derivative of f can be considered a map $df : TM \rightarrow TM$ where $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle of M and $df_x : T_x M \rightarrow T_{f(x)} M$. A closed subset $\Lambda \subset M$ is *hyperbolic* if $f(\Lambda) = \Lambda$ and each tangent space $T_x M$ with $x \in \Lambda$ can be written as a direct sum

$$T_x M = E_x^u \oplus E_x^s$$

of subspaces so that

$$1. \quad Df(E_x^s) = E_{f(x)}^s, Df(E_x^u) = E_{f(x)}^u;$$

2. there exist constants $c > 0$ and $\lambda \in (0, 1)$ so that

$$\|Df^n(v)\| \leq c\lambda^n \|v\| \text{ when } v \in E_x^s, n \geq 0$$

and

$$\|Df^{-n}(v)\| \leq c\lambda^n \|v\| \text{ when } v \in E_x^u, n \geq 0;$$

3. E_x^s, E_x^u vary continuously with x .

Furthermore, we say f satisfies *Axiom A* if $\Omega(f)$ is hyperbolic and the periodic points are dense in $\Omega(f)$. It is well known that the basic set of Axiom A systems is expansive, transitive and satisfies the shadowing property.

2 Preliminaries

2.1 Notions and Notations

Consider a metric space (O, d) . Let A, B be two nonempty subsets, then the distance from $x \in X$ to B is defined as

$$\text{dist}(x, A) := \inf_{y \in A} d(x, y).$$

Furthermore, the distance from A to B is defined as

$$\text{dist}(A, B) := \sup_{x \in A} \text{dist}(x, B).$$

Finally, the Hausdorff distance between A and B is defined as

$$d_H(A, B) := \max \{ \text{dist}(A, B), \text{dist}(B, A) \}.$$

Now consider a TDS (X, f) . If for every pair of non-empty open sets U, V there is an integer n such that $f^n(U) \cap V \neq \emptyset$ then we call (X, f) *topologically transitive*. Furthermore, if for every pair of non-empty open sets U, V there exists an integer N such that $f^n(U) \cap V \neq \emptyset$ for every $n > N$, then we call (X, f) *topologically mixing*. We say that f is *positively expansive* if there exists a constant $c > 0$ such that for any $x, y \in X$, $d(f^i x, f^i y) > c$ for some $i \in \mathbb{Z}^+$. When f is a homeomorphism, we say that f is *expansive* if there exists a constant $c > 0$ such that for any $x, y \in X$, $d(f^i x, f^i y) > c$ for some $i \in \mathbb{Z}$. We call c the expansive constant.

Fix an arbitrary $x \in X$. Let $V_\epsilon(x)$ denote a ball centered at x with radius ϵ . Then it is easy to check that

$$\begin{aligned} x \in AP(f) &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \text{ is syndetic} \Leftrightarrow \forall \epsilon > 0, B_*(N(x, V_\epsilon(x))) > 0, \\ x \in Rec(f) &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \neq \emptyset, \\ x \in \Omega(f) &\Leftrightarrow \forall \epsilon > 0, N(V_\epsilon(x), V_\epsilon(x)) \neq \emptyset. \end{aligned}$$

Other kind of recurrence such as weak almost periodic, quasi-weak almost periodic and Banach recurrent have been discussed in [38, 74]. We say $(Y, f|_Y)$ is a subsystem of (X, f) if Y is a closed f -invariant subset of X and $f|_Y$ is the restriction of f on Y . It is not hard to check that $Rec(f|_Y) = Rec(f) \cap Y$. Consequently, $NR(f|_Y) = NR(f) \cap Y$.

A finite sequence $\mathfrak{C} = \langle x_1, \dots, x_l \rangle, l \in \mathbb{N}$ is called a *chain*. Furthermore, if $d(fx_i, x_{i+1}) < \varepsilon, 1 \leq i \leq l-1$, we call \mathfrak{C} an ε -chain. For any $m \in \mathbb{N}$, if there are m chains $\mathfrak{C}_i = \langle x_{i,1}, \dots, x_{i,l_i} \rangle, l_i \in \mathbb{N}, 1 \leq i \leq m$ satisfying that $x_{i,l_i} = x_{i+1,1}, 1 \leq i \leq m-1$, then can concatenate \mathfrak{C}_i s to constitute a new chain

$$\langle x_{1,1}, \dots, x_{1,l_1}, x_{2,2}, \dots, x_{2,l_2}, \dots, x_{m,2}, \dots, x_{m,l_m} \rangle$$

which we denote by $\mathfrak{C}_1 \mathfrak{C}_2 \dots \mathfrak{C}_m$. When $m = \infty$, we can also concatenate \mathfrak{C}_i s to obtain a pseudo-orbit

$$\langle x_{1,1}, \dots, x_{1,l_1}, x_{2,2}, \dots, x_{2,l_2}, x_{3,2}, \dots, x_{3,l_3}, \dots \rangle$$

which we denote by $\mathfrak{C}_1 \mathfrak{C}_2 \dots \mathfrak{C}_m \dots$.

Let $A \subseteq X$ be a nonempty invariant set. We call A *internally chain transitive* if for any $a, b \in A$ and any $\varepsilon > 0$, there is an ε -chain \mathfrak{C}_{ab} in A with connecting a and b .

For any two TDSs (X, f) and (Y, g) , if $\pi : (X, f) \rightarrow (Y, g)$ is a continuous surjection such that $\pi \circ f = g \circ \pi$, then we say π is a semiconjugation. We have the following conclusions:

$$\pi(AP(f)) = AP(g) \text{ and } \pi(Rec(f)) = Rec(g). \quad (2.2)$$

If in addition, π is a homeomorphism, we call π a conjugation.

2.2 The Space of Borel Probability Measures

2.2.1 Metric Compatible with the Weak* Topology

The space of Borel probability measures on X is denoted by $M(X)$ and the set of continuous functions on X by $C(X)$. We endow $\varphi \in C(X)$ the norm $\|\varphi\| = \max\{|\varphi(x)| : x \in X\}$. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a dense subset of $C(X)$, then

$$\rho(\xi, \tau) = \sum_{j=1}^{\infty} \frac{|\int \varphi_j d\xi - \int \varphi_j d\tau|}{2^j \|\varphi_j\|}$$

defines a metric on $M(X)$ for the *weak** topology [75].

Proposition 2.1. [75] Let $\mu_n, \mu \in M(X)$. Then the following conditions are equivalent:

1. μ_n converges weakly to μ .
2. $\lim_{n \rightarrow \infty} \int \varphi d\mu_n = \int \varphi d\mu$ for all $\varphi \in C(X)$.
3. $\limsup_{n \rightarrow \infty} \mu_n(C) \leq \mu(C)$ for all closed $C \subseteq X$.
4. $\liminf_{n \rightarrow \infty} \mu_n(U) \geq \mu(U)$ for all open $U \subseteq X$.

For $\nu \in M(X)$ and $r > 0$, we denote a ball in $M(X)$ centered at ν with radius r by

$$\mathcal{B}(\nu, r) := \{\rho(\nu, \mu) < r : \mu \in M(X)\}.$$

One notices that

$$\rho(\xi, \tau) \leq 2 \text{ for any } \xi, \tau \in M(X). \quad (2.3)$$

It is also well known that the natural imbedding $j : x \mapsto \delta_x$ is continuous. Since X is compact and $M(X)$ is Hausdorff, one sees that there is a homeomorphism between X and its image $j(X)$. Therefore, without loss of generality we will assume that

$$d(x, y) = \rho(\delta_x, \delta_y). \quad (2.4)$$

A straight calculation using (2.3) and (2.4) gives

Lemma 2.2. Let (X, f) be a dynamical system and let $x \in X$.

1. Let $0 \leq k < m < n$ and $x \in X$. Then

$$\rho(\mathcal{E}_m(x), \mathcal{E}_n(f^k(x))) \leq \frac{2}{n}(n - m + k),$$

2. Given $\varepsilon > 0$ and $p \in \mathbb{N}$, for every $y \in B_p(x, \varepsilon)$ we have $\rho(\mathcal{E}_p(y), \mathcal{E}_p(x)) < \varepsilon$.

3. Given $\varepsilon > 0$ and $p, q \in \mathbb{N}$ satisfying $p \leq q \leq (1 + \varepsilon/2)p$, for every $y \in B_p(x, \varepsilon)$ we have $\rho(\mathcal{E}_q(y), \mathcal{E}_p(x)) < 2\varepsilon$.

Definition 2.3. For $\mu \in M(X)$, the set of all $x \in X$ with the property that $\mu(U) > 0$ for all neighborhood U of x is called the support of μ and denoted by S_μ . Alternatively, S_μ is the (well defined) smallest closed set C with $\mu(C) = 1$.

2.2.2 The Space of Invariant Measures

We say $\mu \in M(X)$ is an f -invariant measure if for any Borel measurable set A , one has $\mu(A) = \mu(f^{-1}A)$. The set of f -invariant measures are denoted by $M(f, X)$. We remark that if $\mu \in M(f, X)$, then S_μ is a closed f -invariant set.

Let (X, f) and (Y, g) be two dynamical systems and $\pi : X \rightarrow Y$ is a continuous map. Define $\pi_* : M(X) \rightarrow M(Y)$ as $\pi_*\mu := \mu \circ \pi^{-1}$ and $\pi^* : C(Y) \rightarrow C(X)$ as $\pi^*\varphi := \varphi \circ \pi$. We call $\pi_*\mu$ the push-forward of μ and $\pi^*\varphi$ the pull-back of φ . It is not hard to see that

$$\int \varphi d\pi_*\mu = \int \pi^*\varphi d\mu \text{ for any } \varphi \in C(Y) \text{ and } \mu \in M(X). \quad (2.5)$$

Moreover, one notices that $\mu \in M(f, X)$ is equivalent to $f_*\mu = \mu$.

The following are not hard to check.

Lemma 2.4. If $\pi : (X, f) \rightarrow (Y, g)$ is a conjugation, then

- (1) there is a metric ρ_X on $M(X)$ and a metric ρ_Y on $M(Y)$ such that $\pi_* : (M(X), \rho_X) \rightarrow (M(Y), \rho_Y)$ is an isometric isomorphism. In particular, $\pi_* : M(X) \rightarrow M(Y)$ is a homeomorphism.
- (2) $\pi_*(M(f, X)) = M(g, Y)$ and $\pi_*|_{M(f, X)} : (M(f, X), \rho_X) \rightarrow (M(f, Y), \rho_Y)$ is also an isometry isomorphism.

We say $\mu \in M(X)$ is an ergodic measure if for any Borel set B with $f^{-1}B = B$, either $\mu(B) = 0$ or $\mu(B) = 1$. We denote the set of ergodic measures on X by $M_{erg}(f, X)$. It is well known that the ergodic measures are exactly the extreme points of $M(f, X)$. We have the following observation.

Lemma 2.5. Let Λ, Λ_0 be three closed f -invariant subset of X with $\Lambda_0 \subseteq \Lambda$. Suppose for any $x \in \Lambda$, $\omega_f(x) \subseteq \Lambda_0$. Then

$$M(f, \Lambda) = M(f, \Lambda_0).$$

Proof. Of course, $M(f, \Lambda) \supseteq M(f, \Lambda_0)$. We now prove that $M(f, \Lambda) \subseteq M(f, \Lambda_0)$. In fact, by the convexity of $M(f, \Lambda)$ and $M(f, \Lambda_0)$, it is sufficient to prove that for any $\mu \in M_{erg}(f, \Lambda)$, $\mu \in M(f, \Lambda_0)$. Indeed, choose an arbitrary generic point $x \in \Lambda$ of μ . Then $S_\mu \subseteq \omega_f(x) \subseteq \Lambda_0$ which yields that $\mu \in M(f, \Lambda_0)$. The proof is completed. \square

Moreover, we have the following characterization.

Lemma 2.6. If $\mu \in M_{erg}(f, X)$, then for any $n \in \mathbb{N}$, there is a $\nu \in M_{erg}(f^n, X)$ and an $m \in \mathbb{N}$ with $m|n$ such that μ can decompose as

$$\mu = \frac{1}{m}(\nu + f_*\nu + \cdots + f_*^{m-1}\nu).$$

Moreover, there is a $X_0 \subseteq X$ with $\nu(X_0) = 1$ and $f^m X_0 = X_0$ such that X has a mod 0 measurable partition $X = \bigsqcup_{i=0}^{m-1} f^i X_0$.

For any $x \in X$, we define the measure center of x as

$$C_x := \overline{\bigcup_{\mu \in M(f, \omega_f(x))} S_\mu}.$$

Furthermore, we define the measure center of Λ as

$$C_\Lambda := \overline{\bigcup_{\mu \in M(f, \Lambda)} S_\mu}.$$

Lemma 2.7. We have the following relations:

$$(1) \quad \bigcap_{\mu \in M_{erg}(f, \Lambda)} S_\mu = \bigcap_{\mu \in M(f, \Lambda)} S_\mu.$$

$$(2) \quad C_\Lambda = \overline{\bigcup_{\mu \in M_{erg}(f, \Lambda)} S_\mu}.$$

Proof. (1). It is clear that $\bigcap_{\mu \in M_{erg}(f, \Lambda)} S_\mu \supseteq \bigcap_{\mu \in M(f, \Lambda)} S_\mu$. So we only need to prove that $\bigcap_{\mu \in M_{erg}(f, \Lambda)} S_\mu \subseteq \bigcap_{\mu \in M(f, \Lambda)} S_\mu$. Indeed, for any $x \in \bigcap_{\mu \in M_{erg}(f, \Lambda)} S_\mu$ and $\varepsilon > 0$, one has

$$\mu(B(x, \varepsilon)) > 0 \text{ for any } \mu \in M_{erg}(f, \Lambda). \quad (2.6)$$

If $\nu(B(x, \varepsilon)) = 0$ for some $\nu \in M(f, \Lambda)$, then by the ergodic decomposition theorem [75], there is a unique measure τ on the Borel subsets of the compact metrisable space $M(f, \Lambda)$ such that $\tau(M(f, \Lambda)) = 1$ and

$$0 = \nu(B(x, \varepsilon)) = \int_{M(f, \Lambda)} \mu(B(x, \varepsilon)) d\tau(\mu).$$

Therefore, for τ -a.e. $\mu \in M_{erg}(f, \Lambda)$, $\mu(B(x, \varepsilon)) = 0$, contradicting (2.6). Thus $x \in \bigcap_{\mu \in M(f, \Lambda)} S_\mu$ which implies that $\bigcap_{\mu \in M_{erg}(f, \Lambda)} S_\mu \subseteq \bigcap_{\mu \in M(f, \Lambda)} S_\mu$.

(2). It is clear that $\overline{\bigcup_{\mu \in M_{erg}(f, \Lambda)} S_\mu} \subseteq \overline{\bigcup_{\mu \in M(f, \Lambda)} S_\mu}$. So it is sufficient to prove that $\overline{\bigcup_{\mu \in M_{erg}(f, \Lambda)} S_\mu} \supseteq \overline{\bigcup_{\mu \in M(f, \Lambda)} S_\mu}$. Indeed, for any $\mu \in M(f, \Lambda)$ and any $x \in S_\mu$, one has that

$$\mu(B(x, \varepsilon)) > 0 \text{ for any } \varepsilon > 0.$$

By the ergodic decomposition theorem, there is a $\mu_\varepsilon \in M_{erg}(f, \Lambda)$ with $\mu_\varepsilon(B(x, \varepsilon)) > 0$. This implies that $B(x, \varepsilon) \cap S_{\mu_\varepsilon} \neq \emptyset$. Since $\varepsilon > 0$ is arbitrary, $x \in \bigcup_{\mu \in M_{erg}(f, \Lambda)} S_\mu$, which yields that $\overline{\bigcup_{\mu \in M_{erg}(f, \Lambda)} S_\mu} \supseteq \overline{\bigcup_{\mu \in M(f, \Lambda)} S_\mu}$. \square

2.3 Topological Entropy and Metric Entropy

2.3.1 Topological Entropy for Compact Sets

The classical topological entropy for (X, f) was introduced by Adler *et al* [1] using open covers. Later on, Bowen gave a equivalent definition of topological entropy for (X, f) using separated and spanning sets [14] which we shall recall now.

For $x, y \in X$ and $n \in \mathbb{N}$, the Bowen distance between x, y is defined as

$$d_n(x, y) := \max\{d(f^i x, f^i y) : i = 0, 1, \dots, n-1\}$$

and the Bowen ball centered at x with radius $\varepsilon > 0$ is defined as

$$B_n(x, \varepsilon) := \{y \in X : d_n(x, y) < \varepsilon\}.$$

Let $Z \subset X$. A set S is (n, ε) -separated for Z if $S \subset Z$ and $d_n(x, y) > \varepsilon$ for any $x, y \in S$ and $x \neq y$. A set $S \subset Z$ is (n, ε) -spanning for Z if for any $x \in Z$, there exists $y \in S$ such that $d_n(x, y) \leq \varepsilon$.

Define

$$\begin{aligned} s_n(Z, \varepsilon) &= \sup \{|S| : S \text{ is } (n, \varepsilon) \text{ - separated for } Z\}, \\ r_n(Z, \varepsilon) &= \inf \{|S| : S \text{ is } (n, \varepsilon) \text{ - spanning for } Z\}, \end{aligned}$$

where $|S|$ denotes the cardinality of S . It is not hard to see that [75]

$$r_n(Z, \varepsilon) \leq s_n(Z, \varepsilon) \leq r_n(Z, \varepsilon/2). \quad (2.7)$$

Definition 2.8. The topological entropy for a compact set $K \subset X$ is defined (by Bowen) as

$$h_d(f, K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(K, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(K, \varepsilon)$$

and for a general set $E \subset X$ as

$$h_d(f, E) = \sup\{h_d(f, K) : K \subset E \text{ compact}\}.$$

2.3.2 Topological Entropy for Noncompact Set

As for noncompact sets, Bowen also developed a satisfying definition via dimension language [13] which we now illustrate.

Let $E \subseteq X$, and $\mathcal{G}_n(E, \sigma)$ be the collection of all finite or countable covers of E by sets of the form $B_u(x, \sigma)$ with $u \geq n$. We set

$$C(E; t, n, \sigma, f) := \inf_{\mathcal{C} \in \mathcal{G}_n(E, \sigma)} \sum_{B_u(x, \sigma) \in \mathcal{C}} e^{-tu}$$

and

$$C(E; t, \sigma, f) := \lim_{n \rightarrow \infty} C(E; t, n, \sigma, f).$$

Then we define

$$h_{\text{top}}(E; \sigma, f) := \inf\{t : C(E; t, \sigma, f) = 0\} = \sup\{t : C(E; t, \sigma, f) = \infty\}$$

The *Bowen topological entropy* of E is

$$h_{\text{top}}(f, E) := \lim_{\sigma \rightarrow 0} h_{\text{top}}(E; \sigma, f). \quad (2.8)$$

We have the following lemma from Bowen [?].

Lemma 2.1. *Let $f : X \rightarrow X$ be a continuous map on a compact metric space. Set*

$$QR(t) = \{x \in X \mid \exists \tau \in V_f(x) \text{ with } h_\tau(T) \leq t\}.$$

Then

$$h_{\text{top}}(f, QR(t)) \leq t.$$

2.3.3 Metric Entropy

We call (X, \mathcal{B}, μ) a probability space if \mathcal{B} is a Borel σ -algebra on X and μ is a probability measure on X . For a finite measurable partition $\xi = \{A_1, \dots, A_n\}$ of a probability space (X, \mathcal{B}, μ) , define

$$H_\mu(\xi) = - \sum_{i=1}^n \mu(A_i) \log \mu(A_i).$$

Let $f : X \rightarrow X$ be a continuous map preserving μ . We denote by $\bigvee_{i=0}^{n-1} f^{-i}\xi$ the partition whose element is the set $\bigcap_{i=0}^{n-1} f^{-i}A_{j_i}$, $1 \leq j_i \leq n$. Then the following limit exists:

$$h_\mu(f, \xi) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} f^{-i}\xi \right)$$

and we define the metric entropy of μ as

$$h_\mu(f) := \sup \{ h_\mu(f, \xi) : \xi \text{ is a finite measurable partition of } X \}.$$

We have the following Katok's entropy formula.

Lemma 2.9. For any $\nu \in M_{erg}(f, X)$ and any $0 < \gamma < 1$,

$$\begin{aligned} h_\nu(f) &= \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \inf \{ r_n(A, \varepsilon) : \nu(A) \geq \gamma \} \\ &= \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \inf \{ r_n(A, \varepsilon) : \nu(A) \geq \gamma \}. \end{aligned}$$

2.3.4 Characterizing Metric Entropy

For $x \in X$, we define the empirical measure of x as

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)},$$

where δ_x is the Dirac mass at x . Let $F \subseteq M(f, X)$ be a neighborhood. For $n \in \mathbb{N}$, define

$$X_{n,F} := \{x \in X : \mathcal{E}_n(x) \in F\}.$$

Lemma 2.10. [26] Let $\mu \in M(f, X)$ be ergodic with $h_\mu(f) > 0$. Then for any $\eta > 0$, there exists $\varepsilon > 0$ such that for each neighbourhood F of μ in $M(X)$, there exists $n_F \in \mathbb{N}$ such that for any $n \geq n_F$, there exists an (n, ε) -separated set $\Gamma_n \subset X_{n,F} \cap S_\mu$ with

$$|\Gamma_n| \geq 2^{n(h_\mu(f) - \eta)}.$$

2.4 Pseudo-orbit Tracing Properties

2.4.1 Shadowing Property

Definition 2.2. For any $\delta > 0$, a sequence $\{x_n\}_{n=0}^{+\infty}$ is called a δ -pseudo-orbit if

$$d(f(x_n), x_{n+1}) < \delta \text{ for } n \in \mathbb{Z}^+.$$

Furthermore, $\{x_n\}_{n=0}^{+\infty}$ is ε -shadowed by some $y \in X$ if

$$d(f^n(y), x_n) < \varepsilon \text{ for any } n \in \mathbb{Z}^+.$$

Finally, we say that (X, f) has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any δ -pseudo-orbit is ε -shadowed by some point in X .

Lemma 2.11. Suppose (X, f) is topologically expanding and transitive (resp., (X, f) is a homeomorphism that is topologically hyperbolic and transitive). Let $A \subsetneq X$ be a closed invariant subset of X . Then there exists an $x \in X$ such that

$$A \subseteq \omega_f(x) \subseteq \bigcup_{l=0}^{\infty} f^{-l}A. \quad (2.9)$$

In particular,

$$\overline{\bigcup_{y \in \omega_f(x)} \omega_f(y)} \subseteq A \subseteq \omega_f(x) \neq X. \quad (2.10)$$

Proof. Let c be the expansive constant. By the shadowing property, there exists a $\delta > 0$ such that any δ -pseudo orbit can be c -shadowed. Since A is compact, we cover A by finite number of balls $\{B(y_i, \delta)\}_{i=1}^p$. By the transitivity, for any two open balls $B(y_i, \delta)$ and $B(y_j, \delta)$, there exists a $w_{ij} \in B(y_i, \delta)$ such that $f^{l_{ij}}(w_{ij}) \in B(y_j, \delta)$ for some $l_{ij} \in \mathbb{N}$. Let

$$L = \max\{l_{ij} : 1 \leq i, j \leq p\}. \quad (2.11)$$

Meanwhile, as a compact subset of compact metric space, A is separable, i.e. there exist $\{x_n\}_{n \geq 1} \subseteq A$ which is dense in A . Now let us construct the δ -pseudo orbit as follows. In the r th step, there exist an $n \in \mathbb{N}$ and $1 \leq t \leq n$ such that $r = \frac{n(n-1)}{2} + t$. Then let $z_r = x_t$ and choose the orbit segment $\mathfrak{D}_r = \langle z_r, \dots, f^r z_r \rangle$. After that, we concatenate the adjacent orbit segments \mathfrak{D}_r and \mathfrak{D}_{r+1} using the orbit segment $\mathfrak{C}_r = \langle w_r, \dots, f^{l_r}(w_r) \rangle$, where \mathfrak{C}_r is chosen from $\{\langle w_{ij}, \dots, f^{l_{ij}}(w_{ij}) \rangle : 1 \leq i, j \leq p\}$. Hence we obtain the following pseudo orbit:

$$\mathfrak{D} = \mathfrak{D}_1 \mathfrak{C}_1 \cdots \mathfrak{D}_r \mathfrak{C}_r \cdots.$$

It is clear that \mathfrak{D} constitutes an δ -pseudo orbit. So there exists an $x \in B(z_1, c)$ such that x c -shadows \mathfrak{D} . Put $j_1 = k_1 = 0$ and $j_m = k_{m-1} + |\mathfrak{C}_{m-1}|$, $k_m = j_{m-1} + m$ inductively for $m \geq 1$.

We now prove that $A \subseteq \omega_f(x)$. Indeed, it is sufficient to prove that for any n , $x_n \in \omega_f(x)$. For then the denseness of $\{x_n\}_{n \geq 1}$ in A and the closeness of A yield the result. In fact, x_n occurs infinitely in \mathfrak{D} at the place $\{j_{\frac{k(k-1)}{2}+n}\}_{k=n}^{\infty}$. By the compactness of X , we suppose (by taking a subsequence if necessary) that $\lim_{k \rightarrow \infty} f^{j_{\frac{k(k-1)}{2}+n}} x = u \in \omega_f(x)$. Then $d(f^i x_n, f^i u) = \lim_{k \rightarrow \infty} d(f^i x_n, f^{j_{\frac{k(k-1)}{2}+n}+i} x) < c$. By the positive expansiveness, we have $u = x_n$. Hence, $x_n \in \omega_f(x)$.

On the other hand, for any $v \in \omega_f(x)$, there exists a strictly increasing sequence n_s such that $v = \lim_{s \rightarrow \infty} f^{n_s}(x)$. The following discussion splits into two cases.

Case 1. For any $p \geq 1$, there exists an n_{s_p} such that $j_{m_p} \leq n_{s_p} < n_{s_p} + p \leq k_{m_p}$ for some $m_p \in \mathbb{N}$. Suppose (take a subsequence if necessary) $\lim_{p \rightarrow \infty} f^{n_{s_p} - j_{m_p}} z_{j_{m_p}} = a$. Then $a \in A$ since A is invariant and closed. Since $d(f^{n_{s_p}+i} x, f^{n_{s_p} - j_{m_p} + i} z_{j_{m_p}}) < c$ for $0 \leq i \leq p$ and all p . So we have $d(f^i v, f^i a) < c$ for all i . By the positive expansiveness, we have $v = a \in A$. In particular, $\omega_f(v) \subseteq A$.

Case 2. There is \tilde{p} such that for all $s \geq 1$, there exists no $m \geq 1$ such that $j_m \leq n_s < n_s + \tilde{p} \leq k_m$. In other words, for any $s \geq 1$, there is some $m_s \geq \tilde{p}$ such that $k_{m_s} - \tilde{p} < n_s < j_{m_s+1}$. Using pigeonhole principle, we suppose (by taking a subsequence if necessary) that $n_s = j_{m_s+1} - l$ for some $0 < l \leq \tilde{p} + L$ where L is defined in (2.11). Since $d(f^{j_{m_s+1}+i} x, f^i z_{j_{m_s+1}}) < c$ for $0 \leq i \leq m_s + 1$ and all s . By the compactness of A , we suppose (by taking a subsequence if necessary) that $\lim_{s \rightarrow \infty} z_{j_{m_s+1}} = z \in A$. Then we have $d(f^{i+l} v, f^i z) < c$ for all i . By the positive expansiveness, we have $f^l v = z \in A$.

We thus have proved (2.9), which immediately implies (2.10). \square

2.4.2 Limit-shadowing and s-Limit-shadowing

Definition 2.12. A sequence $\{x_i\}_{i=0}^{+\infty}$ is called a limit-pseudo-orbit if

$$\lim_{i \rightarrow \infty} d(f(x_i), x_{i+1}) = 0.$$

Moreover, $\{x_i\}_{i \in \mathbb{N}}$ are limit-shadowed by $y \in X$ if

$$\lim_{i \rightarrow \infty} d(f^i y, x_i) = 0.$$

Then we say (X, f) has the *limit-shadowing* property if any limit-pseudo-orbit can be limit-shadowed.

Definition 2.13. For any $\delta > 0$, a sequence $\{x_i\}_{i=0}^{+\infty}$ is called a δ -limit-pseudo-orbit if $\{x_n\}_{n=0}^{+\infty}$ is both a δ -pseudo-orbit and a limit-pseudo-orbit. Furthermore, $\{x_n\}_{n=0}^{+\infty}$ is ε -limit-shadowed by some $y \in X$ if $\{x_n\}_{n=0}^{+\infty}$ is both ε -shadowed and limit-shadowed by y . Finally, we say that f has the *s-limit-shadowing* property if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any δ -limit-pseudo-orbit can be ε -limit-shadowed.

We have the following result.

Lemma 2.14. [63, 3] If (X, f) is topological expanding (resp. topological hyperbolic), then (X, f) has the s-limit shadowing property.

For convenience, we recall the ω -limit set of a pseudo-orbit $\{x_i\}_{i \in \mathbb{N}}$, given by

$$\omega(\{x_i\}_{i \in \mathbb{N}}) = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{k \geq n} \{x_k\}}.$$

The following lemma is an easy exercise.

Lemma 2.15. If $\{x_i\}_{i \in \mathbb{N}}$ is limit-shadowed by $y \in X$, then $\omega(\{x_i\}_{i \in \mathbb{N}}) = \omega_f(y)$.

2.4.3 Specification and Almost Specification Properties

Definition 2.16. We say that (X, f) satisfies the *specification* property if for all $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that for any collection $\{I_j = [a_j, b_j] \subseteq \mathbb{Z}^+ : j = 1, \dots, k\}$ of finite intervals with $a_{j+1} - b_j \geq m(\varepsilon)$ for $j = 1, \dots, k-1$ and any x_1, \dots, x_k in X , there exists a point $x \in X$ such that

$$d(f^{a_j+t} x, f^t x_j) < \varepsilon$$

for all $t = 0, \dots, b_j - a_j$ and $j = 1, \dots, k$.

Pfister and Sullivan generalized the specification property to the g -approximate product property in the study of large deviation [57]. Later on, Thompson renamed it as the almost specification property in the study of irregular points [73]. The only difference is that the blowup function g can depend on ε in the latter case. However, this subtle difference does not affect our discussion here.

Definition 2.17. Let $\varepsilon_0 > 0$. A function $g : \mathbb{N} \times (0, \varepsilon_0) \rightarrow \mathbb{N}$ is called a *mistake function* if for all $\varepsilon \in (0, \varepsilon_0)$ and all $n \in \mathbb{N}$, $g(n, \varepsilon) \leq g(n+1, \varepsilon)$ and

$$\lim_n \frac{g(n, \varepsilon)}{n} = 0.$$

If $\varepsilon \geq \varepsilon_0$, we define $g(n, \varepsilon) = g(n, \varepsilon_0)$. For $n \in \mathbb{N}$ large enough such that $g(n, \varepsilon) < n$, let $\Lambda_n = \{0, 1, \dots, n-1\}$. Define the $(g; n, \varepsilon)$ -Bowen ball centered at x as the closed set

$$B_n(g; x, \varepsilon) := \{y \in X \mid \exists \Lambda \subseteq \Lambda_n, |\Lambda_n \setminus \Lambda| \leq g(n, \varepsilon) \text{ and } \max\{f^j x, f^j y : j \in \Lambda\} \leq \varepsilon\}.$$

Definition 2.18. The dynamical system (X, f) has the *almost specification* property with mistake function g , if there exists a function $k_g : (0, +\infty) \rightarrow \mathbb{N}$ such that for any $\varepsilon_1 > 0, \dots, \varepsilon_m > 0$, any points $x_1, \dots, x_m \in X$, and any integers $n_1 \geq k_g(\varepsilon_1), \dots, n_m \geq k_g(\varepsilon_m)$, we can find a point $z \in X$ such that

$$f^{l_j}(z) \in B_{n_j}(g; x_j, \varepsilon_j), \quad j = 1, \dots, m,$$

where $n_0 = 0$ and $l_j = \sum_{s=0}^{j-1} n_s$.

2.5 Uniform Separation Property

For $\delta > 0$, $\varepsilon > 0$ and $n \in \mathbb{N}$, two points x and y are (δ, n, ε) -separated if

$$|\{j : d(f^j(x), f^j(y)) > \varepsilon, \quad 0 \leq j \leq n-1\}| \geq \delta n.$$

A subset E is (δ, n, ε) -separated if any pair of different points of E are (δ, n, ε) -separated.

Lemma 2.19. [57] Let $\nu \in M(X)$ be ergodic and $h^* < h_\nu$. Then there exist $\delta^* > 0$ and $\varepsilon^* > 0$ so that for each neighborhood F of ν in $M(X)$, there exists $n_F^* \in \mathbb{N}$ such that for any $n \geq n_F^*$, there exists $\Gamma_n \subseteq X_{n,F}$ which is $(\delta^*, n, \varepsilon^*)$ -separated and satisfies $\log |\Gamma_n| \geq nh^*$.

Furthermore, if the above δ^* and ε^* can be chosen to be independent of μ , we have the following definition.

Definition 2.20. We say (X, f) has the uniform separation property if the following holds. For any $\eta > 0$, there exist $\delta^* > 0$ and $\varepsilon^* > 0$ so that for μ ergodic and any neighbourhood $F \subseteq M(X)$ of μ , there exists $n_{F,\mu,\eta}^* \in \mathbb{N}$ such that for any $n \geq n_{F,\mu,\eta}^*$, there is a $(\delta^*, n, \varepsilon^*)$ -separated set $\Gamma_n \subseteq X_{n,F}$ with

$$|\Gamma_n| \geq 2^{n(h_\mu - \eta)}.$$

Note 2.21. The key observation of the above definition is that the selection of δ^*, ε^* does not depend on μ and F . This is exactly what ‘uniform’ means.

Uniform separation property is satisfied by some typical systems, as the following result shows.

Proposition 2.22. [58] If (X, f) is expansive, h -expansive or asymptotic h -expansive, then (X, f) has the uniform separation property.

For a fixed $\delta > 0$, when n is large enough, a (δ, n, ε) -separated set is also an (n, ε) -separated set. So we have the following lemma.

Lemma 2.23. If (X, f) has the uniform separation property, then for any $\eta > 0$, there exists $\tilde{\varepsilon} > 0$ so that for μ ergodic and any neighbourhood $F \subseteq M(X)$ of μ , there exists $\tilde{n}_{F,\mu,\eta} \in \mathbb{N}$ such that for any $n \geq \tilde{n}_{F,\mu,\eta}$, there is a $(n, \tilde{\varepsilon})$ -separated set $\Gamma_n \subseteq X_{n,F}$ with

$$|\Gamma_n| \geq 2^{n(h_\mu - \eta)}.$$

Furthermore, we use a discussion similar to [58, Corollary 3.1] and obtain the following result.

Proposition 2.24. Assume that (X, f) satisfies the entropy dense and uniform separation property. Then for any $\zeta > 0$, there exist $\varepsilon > 0$ so that for any $\nu \in M(f, X)$ and its neighbourhood $G \subseteq M(X)$, there exists $n_{G,\nu,\zeta} \in \mathbb{N}$ such that for any $n \geq n_{G,\nu,\zeta}$, there is a (n, ε) -separated set $\Gamma_n \subseteq X_{n,G}$ with

$$|\Gamma_n| \geq 2^{n(h_\nu - \zeta)}.$$

3 Density

Now we discuss various density and their basic properties. For any two positive integers $a_k < b_k$, denote $[a_k, b_k] = \{a_k, a_k + 1, \dots, b_k\}$ and $[a_k, b_k) = [a_k, b_k - 1]$, $(a_k, b_k) = [a_k + 1, b_k - 1]$, $(a_k, b_k] = [a_k + 1, b_k]$. A point x is called *quasi-generic* for some measure μ , if there is a sequence of positive integer intervals $I_k = [a_k, b_k)$ with $b_k - a_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \delta_{f^j(x)} = \mu$$

in weak* topology. Let $V_f^*(x) = \{\mu \in M(f, X) : x \text{ is quasi-generic for } \mu\}$. This concept is from [33] and from there it is known $V_f^*(x)$ is always nonempty, compact and connected. Note that $V_f(x) \subseteq V_f^*(x)$.

Lemma 3.1. For (X, f) and $x \in X$, $z \in X_{B_*}(x)$ if and only if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any N consecutive positive integers $n + 1, \dots, n + N$, $d(f^{n+i}x, z) < \varepsilon$ for some $1 \leq i \leq N$.

Proof. Sufficiency: Let $I_k = [a_k, b_k)$ be a sequence of positive integer intervals with $b_k - a_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I_k|}{|I_k|} = \liminf_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I|}{|I|}.$$

Let $b_k - a_k = l_k N + r_k$, $0 \leq r_k \leq N - 1$ for each $k \in \mathbb{N}$. Then for k large enough such that $l_k \geq 1$, one has

$$\frac{|N(x, V_\varepsilon(z)) \cap I_k|}{|I_k|} \geq \frac{l_k}{l_k N + r_k} \geq \frac{1}{2N}.$$

This implies that

$$\liminf_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I|}{|I|} = \lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I_k|}{|I_k|} \geq \frac{1}{2N} > 0.$$

Since ε is arbitrary, we see that $z \in X_{B_*}(x)$.

Necessity: Otherwise, for any $\varepsilon > 0$ and for any $k \in \mathbb{N}$, there exists an $n_k \in \mathbb{N}$ such that $d(f^{n_k+i}x, z) \geq \varepsilon$ for $1 \leq i \leq k$. Now consider the sequence of intervals $I_k = [n_k + 1, n_k + k]$. Then for any $k \in \mathbb{N}$,

$$|N(x, V_\varepsilon(z)) \cap I_k| = 0.$$

This implies that

$$\liminf_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I|}{|I|} = \lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(z)) \cap I_k|}{|I_k|} = 0,$$

contradicting the fact that $z \in X_{B_*}(x)$. □

Corollary 3.2. Fix any $x \in X$. Then for any $y \in \omega_f(x)$, $X_{B_*}(x) \subseteq \overline{\text{orb}(y, f)}$. As a result, $X_{B_*}(x)$ is either empty or a minimal set.

Proof. If $X_{B_*}(x) = \emptyset$, there is nothing to prove. So we suppose $X_{B_*}(x) \neq \emptyset$. Select an arbitrary $z \in X_{B_*}(x)$. For any $y \in \omega_f(x)$, if $z \notin \overline{\text{orb}(y, f)}$, then let $2\varepsilon := \text{dist}(z, \overline{\text{orb}(y, f)}) > 0$. By lemma 3.1, there is an $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $d(f^{n+i}x, z) < \varepsilon$ for some $1 \leq i \leq N$. Now choose a $0 < \delta < \varepsilon$ such that $d(u, v) < \delta$ implies that $d(f^j u, f^j v) < \varepsilon$, $1 \leq j \leq N$. Moreover, since $y \in \omega_f(x)$, we select a $p \in \mathbb{N}$ such that $d(f^p x, y) < \delta$. Then there exist a $1 \leq q \leq N$ such that $d(f^{p+q}x, z) < \varepsilon$. However, the selection of δ and the condition $1 \leq q \leq N$ indicate that $d(f^{p+q}x, f^q y) < \varepsilon$. So we have $d(z, f^q y) < 2\varepsilon$, contradicting the definition of ε . Therefore, $z \in \overline{\text{orb}(y, f)}$. Since $z \in X_{B_*}(x)$ is arbitrary, we have $X_{B_*}(x) \subseteq \overline{\text{orb}(y, f)}$.

Moreover, it is well known that (by Zorn lemma) every topological dynamical system has a minimal system. So we select some z from a minimal subset of $\omega_f(x)$. Therefore, $\overline{\text{orb}(z, f)}$ is minimal. Moreover, since $X_{B_*}(x) \subseteq \overline{\text{orb}(z, f)}$ and $X_{B_*}(x)$ is nonempty, closed and f -invariant, $X_{B_*}(x)$ is minimal, which completes the proof. \square

Further, by (1.1), we see that for any $x \in X$,

$$\begin{aligned} X_{B_*}(x) &= \omega_f(x) \\ \Leftrightarrow X_{B_*}(x) &= X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x) \\ \Rightarrow \omega_f(x) &\text{ is minimal.} \end{aligned}$$

Furthermore, we have the following characterization of the $\xi - \omega$ -limit set, $\xi = \overline{d}, \underline{d}, B^*, B_*$.

Proposition 3.3. For any $x \in X$,

- (1) $X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu$;
- (2) $X_{\overline{d}}(x) = \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$;
- (3) $X_{B_*}(x) = \bigcap_{\mu \in V_f^*(x)} S_\mu$;
- (4) $X_{B^*}(x) = \overline{\bigcup_{\mu \in V_f^*(x)} S_\mu}$.

Proof. (1) On one hand, consider an arbitrary $y \in X_{\underline{d}}(x)$. For any $\mu \in V_f(x)$, let $m_k \rightarrow \infty$ be such that $\lim_{k \rightarrow \infty} \mathcal{E}_{m_k}(x) = \mu$. Then for any $\varepsilon > 0$, one has

$$\begin{aligned} \mu(V_{2\varepsilon}(y)) &\geq \mu(\overline{V_\varepsilon(y)}) \geq \limsup_{k \rightarrow \infty} \mathcal{E}_{m_k}(\overline{V_\varepsilon(y)}) \\ &= \limsup_{k \rightarrow \infty} \frac{1}{m_k} \sum_{i=0}^{m_k-1} \delta_{f^i x}(\overline{V_\varepsilon(y)}) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(\overline{V_\varepsilon(y)}) > 0. \end{aligned}$$

This implies that $y \in S_\mu$ and thus $X_{\underline{d}}(x) \subseteq \bigcap_{\mu \in V_f(x)} S_\mu$.

On the other hand, consider an arbitrary $y \in \bigcap_{\mu \in V_f(x)} S_\mu$. For any $\varepsilon > 0$, let $n_k \rightarrow \infty$ be such that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(V_\varepsilon(y)) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(V_\varepsilon(y)).$$

Then choose a subsequence n_{k_l} of n_k such that $\lim_{l \rightarrow \infty} \mathcal{E}_{n_{k_l}}(x) = \tau$ for some $\tau \in V_f(x)$. Then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(V_\varepsilon(y)) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(V_\varepsilon(y)) = \lim_{l \rightarrow \infty} \frac{1}{n_{k_l}} \sum_{i=0}^{n_{k_l}-1} \delta_{f^i x}(V_\varepsilon(y)) \geq \tau(V_\varepsilon(y)) > 0.$$

Therefore, $y \in X_{\underline{d}}(x)$ and thus $X_{\underline{d}}(x) \supseteq \bigcap_{\mu \in V_f(x)} S_\mu$.

(2) On one hand, consider an arbitrary $y \in X_{\underline{d}}(x)$. Then for any $\varepsilon > 0$, one has

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}(V_\varepsilon(y)) > 0.$$

Now choose a sequence $n_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(V_\varepsilon(y)) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(V_\varepsilon(y)).$$

Then choose a subsequence n_{k_l} of n_k such that $\lim_{l \rightarrow \infty} \mathcal{E}_{n_{k_l}}(x) = \tau$ for some $\tau \in V_f(x)$. Then

$$\begin{aligned} \tau(V_{2\varepsilon}(y)) \geq \tau(\overline{V_\varepsilon(y)}) &\geq \lim_{l \rightarrow \infty} \frac{1}{n_{k_l}} \sum_{i=0}^{n_{k_l}-1} \delta_{f^i x}(\overline{V_\varepsilon(y)}) \\ &= \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(\overline{V_\varepsilon(y)}) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(\overline{V_\varepsilon(y)}) > 0. \end{aligned}$$

Therefore, $V_{2\varepsilon}(y) \cap S_\tau \neq \emptyset$. So for each $k \in \mathbb{N}$, we obtain a $y_k \in V_{1/k}(y) \cap S_{\mu_k}$ for some $\mu_k \in V_f(x)$. Thus $y_k \rightarrow y$ as $k \rightarrow \infty$, which implies that $y \in \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$ and thus $X_{\overline{d}}(x) \subseteq \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$.

On the other hand, consider an arbitrary $y \in \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$. For each $k \in \mathbb{N}$, choose a $y_k \in V_{1/k}(y) \cap S_{\mu_k}$ for some $\mu_k \in V_f(x)$. Then $V_{1/k}(y)$ is a neighborhood of $y_k \in S_{\mu_k}$. This implies that $\mu_k(V_{1/k}(y)) > 0$. Since $\mu_k \in V_f(x)$, we choose a sequence $k_l \rightarrow \infty$ such that $\lim_{l \rightarrow \infty} \mathcal{E}_{k_l}(x) = \mu_k$. Then one sees that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}(V_{1/k_l}(y)) \geq \lim_{l \rightarrow \infty} \frac{1}{k_l} \sum_{i=0}^{k_l-1} \delta_{f^i x}(V_{1/k_l}(y)) \geq \mu_k(V_{1/k_l}(y)) > 0.$$

Since $k \in \mathbb{N}$ is arbitrary, one has $y \in X_{\overline{d}}(x)$. Since $y \in \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$ is arbitrary, we see $X_{\overline{d}}(x) \supseteq \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$.

- (3) On one hand, consider an arbitrary $y \in X_{B_*}(x)$. For any $\mu \in V_f^*(x)$, let $I_k = [a_k, b_k]$ be a sequence of positive integer intervals with $\lim_{k \rightarrow \infty} b_k - a_k = \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \delta_{f^j x} = \mu.$$

Then for any $\varepsilon > 0$, one has

$$\begin{aligned} \mu(V_{2\varepsilon}(y)) \geq \mu(\overline{V_\varepsilon(y)}) &= \lim_{k \rightarrow \infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \delta_{f^j x}(\overline{V_\varepsilon(y)}) \\ &= \lim_{k \rightarrow \infty} \frac{|N(x, \overline{V_\varepsilon(y)}) \cap I_k|}{|I_k|} \\ &\geq \liminf_{|I| \rightarrow \infty} \frac{|N(x, \overline{V_\varepsilon(y)}) \cap I|}{|I|} > 0. \end{aligned}$$

This implies that $y \in S_\mu$ and thus $X_{B_*}(x) \subseteq \bigcap_{\mu \in V_f^*(x)} S_\mu$.

On the other hand, consider an arbitrary $y \in \bigcap_{\mu \in V_f^*(x)} S_\mu$. Let $I_k = [a_k, b_k]$ be a sequences of positive integer intervals with $\lim_{k \rightarrow \infty} b_k - a_k = \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I_k|}{|I_k|} = \liminf_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I|}{|I|}.$$

Then choose a subsequence I_{k_l} of I_k such that $\lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)} = \tau$ for some $\tau \in V_f^*(x)$. We have

$$\begin{aligned} \liminf_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I|}{|I|} &= \lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I_k|}{|I_k|} \\ &= \lim_{l \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I_{k_l}|}{|I_{k_l}|} \\ &= \lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)}(V_\varepsilon(y)) \\ &\geq \tau(V_\varepsilon(y)) > 0. \end{aligned}$$

Therefore, $y \in X_{B^*}(x)$ and thus $X_{B^*}(x) \supseteq \bigcap_{\mu \in V_f^*(x)} S_\mu$.

(4) On one hand, consider an arbitrary $y \in X_{B^*}(x)$. Then for any $\varepsilon > 0$, one has

$$\limsup_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I|}{|I|} > 0.$$

Now choose a sequence of positive integer intervals $I_k = [a_k, b_k)$ with $b_k - a_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I_k|}{|I_k|} = \limsup_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I|}{|I|}.$$

Then choose a subsequence I_{k_l} of I_k such that $\lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)} = \tau$ for some $\tau \in V_f^*(x)$. We have

$$\begin{aligned} \tau(V_{2\varepsilon}(y)) \geq \tau(\overline{V_\varepsilon(y)}) &= \lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)}(\overline{V_\varepsilon(y)}) \\ &= \lim_{k \rightarrow \infty} \frac{1}{b_k - a_k} \sum_{i=a_k}^{b_k-1} \delta_{f^i(x)}(\overline{V_\varepsilon(y)}) \\ &= \lim_{k \rightarrow \infty} \frac{|N(x, \overline{V_\varepsilon(y)}) \cap I_k|}{|I_k|} \\ &= \limsup_{|I| \rightarrow \infty} \frac{|N(x, \overline{V_\varepsilon(y)}) \cap I|}{|I|} > 0. \end{aligned}$$

Therefore, $V_\varepsilon(y) \cap S_\tau \neq \emptyset$. So for each $k \in \mathbb{N}$, we obtain a $y_k \in V_{1/k}(y) \cap S_{\mu_k}$ for some $\mu_k \in V_f(x)$. Thus $y_k \rightarrow y$ as $k \rightarrow \infty$, which implies that $y \in \overline{\bigcup_{\mu \in V_f(x)} S_\mu}$ and thus $X_{B^*}(x) \subseteq \overline{\bigcup_{\mu \in V_f^*(x)} S_\mu}$.

On the other hand, consider an arbitrary $y \in \overline{\bigcup_{\mu \in V_f^*(x)} S_\mu}$. For each $k \in \mathbb{N}$, choose a $y_k \in V_{1/k}(y) \cap S_{\mu_k}$ for some $\mu_k \in V_f^*(x)$. Then $V_{1/k}(y)$ is a neighborhood of $y_k \in S_{\mu_k}$. This implies that $\mu_k(V_{1/k}(y)) > 0$. Since $\mu_k \in V_f^*(x)$, we choose a sequence $k_l \rightarrow \infty$ such that

$$\lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)} = \mu_k.$$

Then one sees that

$$\begin{aligned} \limsup_{|I| \rightarrow \infty} \frac{|N(x, V_{1/k}(y)) \cap I|}{|I|} &\geq \lim_{l \rightarrow \infty} \frac{|N(x, V_{1/k}(y)) \cap I_{k_l}|}{|I_{k_l}|} \\ &= \lim_{l \rightarrow \infty} \frac{1}{b_{k_l} - a_{k_l}} \sum_{j=a_{k_l}}^{b_{k_l}-1} \delta_{f^j(x)}(V_{1/k}(y)) \\ &\geq \mu_k(V_{1/k}(y)) > 0. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, let $k \in \mathbb{N}$ be such that $0 < 1/k < \varepsilon$, we see that

$$\limsup_{|I| \rightarrow \infty} \frac{|N(x, V_\varepsilon(y)) \cap I|}{|I|} \geq \limsup_{|I| \rightarrow \infty} \frac{|N(x, V_{1/k}(y)) \cap I|}{|I|} > 0$$

This indicates that $y \in X_{B^*}(x)$ and thus $X_{B^*}(x) \supseteq \overline{\bigcup_{\mu \in V_f^*(x)} S_\mu}$. □

Corollary 3.4. For any $x \in X$, $X_{B^*}(x) = \bigcap_{\mu \in M(f, \omega_f(x))} S_\mu$ and $X_{B^*}(x) = C_x$.

Proof. Note that $M_{erg}(f, \omega_f(x)) \subseteq V_f^*(x) \subseteq M(f, \omega_f(x))$. Moreover, proposition 3.3 above yields that $X_{B^*}(x) = \bigcap_{\mu \in V_f^*(x)} S_\mu$ and $X_{B^*}(x) = \overline{\bigcup_{\mu \in V_f^*(x)} S_\mu}$. In addition, we recall lemma 2.7 and obtain the required conclusions. □

Remark that (from Proposition 3.3) $X_{B^*}(x)$ and $X_{\underline{d}}(x)$ are admitted empty but $X_{\overline{d}}(x)$ and $X_{B^*}(x)$ are always non-empty.

Proposition 3.5. For any invariant measure μ and μ a.e. $x \in X$,

$$X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x).$$

Proof By ergodic decomposition theorem, we only prove the case that μ is ergodic. By ergodicity, for μ a.e. $x \in X$, any $y \in S_\mu$ and any $\epsilon > 0$, $N(x, V_\epsilon(y))$ has positive density (equal to $\mu(V_\epsilon(y))$) w.r.t. \underline{d} and \overline{d} . So for μ a.e. $x \in X$, $S_\mu \subseteq X_{\underline{d}}(x)$. By ergodicity, we also have that for μ a.e. $x \in X$, $S_\mu = \omega_f(x)$. So we conclude the result. □

Proposition 3.6. For any $x \in X$, $M_{erg}(f, \omega_f(x)) \subseteq V_f^*(x)$ and

$$\overline{\bigcup_{\mu \in M_{erg}(f, \omega_f(x))} S_\mu} \subseteq X_{B^*}(x).$$

If $M_{erg}(f, \omega_f(x))$ is dense in $M(f, \omega_f(x))$, then $V_f^*(x) = M(f, \omega_f(x))$. If further f has an invariant measure μ with $S_\mu = \omega_f(x)$, then $X_{B^*}(x) = \omega_f(x)$.

Proof. From [33, Proposition 3.9, Page 65] we know that for a point x_0 and an ergodic measure $\mu_0 \in M(f, \omega_f(x_0))$, x_0 is quasi-generic for μ_0 . By assumption of density of ergodic measures, $V_f^*(x) = M(X, \omega_f(x_0))$. If further f has an invariant measure μ with $S_\mu = \omega_f(x)$, then from Proposition 3.3 we know $S_\mu \subseteq X_{B^*}(x)$ so that $X_{B^*}(x) = \omega_f(x)$. □

4 Entropy-dense Properties

Eizenberg, Kifer and Weiss proved for systems with the specification property that [28] any f -invariant probability measure ν is the weak limit of a sequence of ergodic measures $\{\nu_n\}$, such that the entropy of ν is the limit of the entropies of the $\{\nu_n\}$. This is a central point in large deviations theory, which was first emphasized in [32]. Meanwhile, this also plays an crucial part in the computing of Billingsley dimension [8, 9] on shift spaces [56]. Pfister and Sullivan refer to this property as the *entropy-dense* property [57].

In this subsection, we introduce various other entropy-dense properties, which shall serve for our needs in the future. In particular, we show that they are satisfied for some typical systems.

4.1 Auxiliary Results

Proposition 4.1. Consider $\Delta \subseteq X$ which satisfies $f^k \Delta = \Delta$ for some $k \in \mathbb{N}$ and let $\Lambda = \bigcup_{i=0}^{k-1} f^i \Delta$. Then

1. if (Δ, f^n) is expansive, then (Λ, f) is expansive;
2. if (Δ, f^n) is topologically transitive, then (Λ, f) is topologically transitive;
3. if (Δ, f^n) has the shadowing property, then (Λ, f) also has the shadowing property.

Proof. It is not hard to check and we omit the details. \square

Proposition 4.2. Consider $\Delta \subseteq X$ which satisfies $f^k \Delta = \Delta$ for some $k \in \mathbb{N}$ and let $\Lambda = \bigcup_{i=0}^{k-1} f^i \Delta$. If we define $\bar{\nu} = j(\nu) = \frac{1}{k} \sum_{i=0}^{k-1} f_*^i \nu$ for each $\nu \in M(f^k, \Delta)$, then

- (1) $j(M(f^k, \Delta)) \subseteq M(f, \Lambda)$. Moreover, there is a metric ρ_1 on $M(f^k, \Delta)$ and a metric ρ_2 on $M(f, \Lambda)$ such that $j : (M(f^k, \Delta), \rho_1) \rightarrow (M(f, \Lambda), \rho_2)$ is an isometric imbedding.

- (2) Furthermore,

$$\begin{aligned} j : (M_{erg}(f^k, \Delta), \rho_1) &\rightarrow (M_{erg}(f, \Lambda), \rho_2) \\ \nu &\mapsto \bar{\nu} \end{aligned}$$

is an isometric isomorphism.

- (3) $h_\nu(f^k|_\Delta) = k h_{\bar{\nu}}(f)$.

Proof. (1) For any $\nu \in M(f^k, \Delta)$, $f_*^k \nu = (f^k)_* \nu = \nu$. So $f_* \bar{\nu} = \bar{\nu}$, which implies that $j(M(f^k, \Delta)) \subseteq M(f, \Lambda)$. To show that j is injective, let us assume $j(\mu) = j(\nu)$. Note that each $f^i \Delta$ is closed and $f_*^i \nu(f^i \Delta) = 1$. So for each $0 \leq i < j \leq k-1$,

$$f_*^i \nu(f^i \Delta \cap f^j \Delta) = f_*^j \nu(f^i \Delta \cap f^j \Delta) = 0.$$

Therefore, for any Borel set $A \subseteq \Delta$, one has $f_*^l \mu(A) = f_*^l \nu(A) = 0, 1 \leq l \leq k-1$. This implies that $\mu(A) = k j(\mu)(A) = k j(\nu)(A) = \nu(A)$. Since $A \subseteq \Delta$ is arbitrary, $\mu = \nu$. Meanwhile, since f_* is continuous, j is continuous, which implies that j is an imbedding. Finally, $M(f^k, \Delta)$ is compact, so the topology does not depend on the selection of metric on $M(f^k, \Delta)$. This enables us to pick the required ρ_1 and ρ_2 .

- (2) Since j is an isometry, it is sufficient to prove that $j : (M_{erg}(f^k, \Delta), \rho_1) \rightarrow (M_{erg}(f, \Lambda), \rho_2)$ is well-defined and is a surjection. Indeed, for any $A \subseteq \Lambda$ with $f^{-1}A = A$, one has $f^{-k}A = A$ and thus $f^{-k}(A \cap \Delta) = A \cap \Delta$. Moreover, $\nu(A) = f_* \nu(A) = \dots = f_*^{k-1} \nu(A)$. So if $0 < j(\nu)(A) < 1$, one has $0 < \nu(A) < 1$, contradicting the fact that $\nu \in M(f^k, \Delta)$. This proves that $j(M_{erg}(f^k, \Delta)) \subseteq M_{erg}(f, \Lambda)$. Moreover, for any $\bar{\nu} \in M(f, \Lambda)$, by lemma 2.6, there is a $\nu \in M_{erg}(f^k, \Delta)$, an $m|k$ and a $X_0 \subseteq X$ satisfying the properties there. Since $f^k \Delta = \Delta$, $\nu(f^i \Delta) = 1$ for some $0 \leq i \leq k-1$. Therefore, $f_*^{k-i} \nu \in M_{erg}(f^k, \Delta)$ and $j(f_*^{k-i} \nu) = \bar{\nu}$, proving that $j : (M_{erg}(f^k, \Delta), \rho_1) \rightarrow (M_{erg}(f, \Lambda), \rho_2)$ is surjective.

- (3) Note that $h_{f_*^i \nu}(f^k|_{f^i \Delta}) = h_{f_*^i \nu}(f^k|_\Lambda), 0 \leq i \leq k-1$. Moreover,

$$h_\nu(f^k|_\Delta) = h_{f_* \nu}(f^k|_{f \Delta}) = \dots = h_{f_*^{k-1} \nu}(f^k|_{f^{k-1} \Delta}).$$

So

$$k h_{\bar{\nu}}(f) = h_{\bar{\nu}}(f^k) = \frac{1}{k} \sum_{i=0}^{k-1} h_{f_*^i \nu}(f^k|_\Lambda) = \frac{1}{k} \sum_{i=0}^{k-1} h_{f_*^i \nu}(f^k|_{f^i \Delta}) = h_\nu(f^k|_\Delta).$$

\square

We also need the following result whose proof shall be given in the appendix.

Proposition 4.3. Consider a m.f.t.-subshift M and a continuous function φ on M . Let $\{\mu^i\}_{i=1}^k, k \geq 2$ be k distinct ergodic measure on M . Then for any $\eta > 0$ and any

$$0 < \varepsilon < \min\{\rho(\mu^i, \mu^j), \text{dist}(\mu^a, \text{cov}\{\mu^b\}_{1 \leq b \leq k, b \neq a}) : 1 \leq i < j \leq k, 1 \leq a \leq k\}, \quad (4.12)$$

there exists a minimal subsystem $\overline{M} \subseteq M$ supporting exactly k ergodic measures $\{\nu^j\}_{j=1}^k$ such that

1

$$\rho(\nu^j, \mu^j) < \varepsilon/2 \text{ for each } j = 1, \dots, k. \quad (4.13)$$

2 $h_{\nu^j} > h_{\mu^j} - \eta, j = 1, \dots, k.$

Lemma A. Suppose (X, f) satisfies the shadowing property. Let $\Lambda \subseteq X$ be closed f -invariant and internally chain transitive. Then for any $\eta > 0$, for any $\mu \in M(f, \Lambda)$ and its neighborhood F_μ , for any $x \in \Lambda$, there exists an $\varepsilon^* = \varepsilon^*(\eta, \mu) > 0$ such that for any $0 < \varepsilon \leq \varepsilon^*$, for any $N \in \mathbb{N}$, there exist an $n = n(F_\mu, \mu, \eta, \varepsilon) \geq N$ such that for any $p \in \mathbb{N}$, there exists an $(pn, \varepsilon^*/3)$ -separated set Γ_{pn} so that

(a) $\Gamma_{pn} \subseteq X_{pn, F_\mu} \cap B(x, \varepsilon) \cap f^{-pn}B(x, \varepsilon);$

(b) $\frac{\log |\Gamma_{pn}|}{pn} > h_\mu - \eta;$

(c) $d_H(\cup_{i=0}^{pn-1} f^i \Gamma_{pn}, \Lambda) < 2\varepsilon.$

Proof. Since F_μ is a neighborhood of μ , there is an $a > 0$ such that $\mathcal{B}(\mu, a) \subseteq F_\mu$. By the ergodic decomposition theorem, there exists a finite convex combination of ergodic measures $\sum_{i=1}^m c_i \nu_i \in \mathcal{B}(\mu, \frac{a}{4})$ with $h_{\sum_{i=1}^m c_i \nu_i} > h_\mu - \frac{\eta}{3}$. Moreover, by the denseness of the rational numbers, we can choose each $c_i = \frac{b_i}{b}$ with $b_i \in \mathbb{N}$ and $\sum_{i=1}^m b_i = b$.

By lemma 2.10, for each i , there exist $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ such that for any $n \geq n_i$, there exists an (n, ε_i) -separated set $\Gamma_n^{\nu_i} \subseteq X_{n, \mathcal{B}(\nu_i, \frac{a}{4})} \cap \Lambda$ such that $\frac{\log |\Gamma_n^{\nu_i}|}{n} > h_{\nu_i} - \frac{\eta}{3}$. Let $\tilde{\varepsilon} = \min\{\varepsilon_i : 1 \leq i \leq m\}$ and $\varepsilon^* = \min\{\frac{\tilde{\varepsilon}}{3}, \frac{a}{4}\}$. By the shadowing property, for any $0 < \varepsilon \leq \varepsilon^*$, there exists a $0 < \delta < \varepsilon$ such that any δ -pseudo-orbit can be ε -shadowed. Moreover, since Λ is compact, we can cover Λ by finite number of open balls $\{B(x_i, \delta)\}_{i=1}^l$ with $\{x_i\}_{i=1}^l \subseteq \Lambda$. Since Λ is internally chain transitive, for each $1 \leq i \leq m$, there exist a δ -chain \mathfrak{C}_{xx_i} with length l_{0i} connecting x to x_i and a δ -chain $\mathfrak{C}_{x_i x}$ with length l_{i0} connecting x_i to x . Let $L = \max\{l_{0i}, l_{i0} : 1 \leq i \leq l\}$. Now choose $k \in \mathbb{N}$ large enough such that

1.

$$kb_i \geq n_i \text{ for } 1 \leq i \leq m \text{ and } kb \geq N; \quad (4.14)$$

2.

$$\frac{2(m+l)L}{kb + 2(m+l)L} \cdot 2 < \frac{a}{4}; \quad (4.15)$$

3.

$$\frac{kb(h_\mu - \frac{2}{3}\eta) - 2m \log l}{kb + 2(m+l)L} > h_\mu - \eta. \quad (4.16)$$

By (4.14), for each $1 \leq i \leq m$, we obtain a (kb_i, ε_i) -separated set $\Gamma_{kb_i}^{\nu_i} \subseteq X_{kb_i, \mathcal{B}(\nu_i, \frac{a}{4})} \cap \Lambda$ such that $\frac{\log |\Gamma_{kb_i}^{\nu_i}|}{kb_i} > h_{\nu_i} - \frac{\eta}{3}$. Moreover, by the pigeonhole principle, there exist $1 \leq i_1, i_2 \leq l$ such that $\tilde{\Gamma}_{kb_i}^{\nu_i} \subseteq B(x_{i_1}, \delta)$ and $f^{kb_i} \tilde{\Gamma}_{kb_i}^{\nu_i} \subseteq B(x_{i_2}, \delta)$. Note that

$$|\tilde{\Gamma}_{kb_i}^{\nu_i}| \geq \frac{|\Gamma_{kb_i}^{\nu_i}|}{l^2}. \quad (4.17)$$

Now let $\Gamma = \tilde{\Gamma}_{kb_1}^{\nu_1} \times \tilde{\Gamma}_{kb_2}^{\nu_2} \times \cdots \times \tilde{\Gamma}_{kb_m}^{\nu_m}$ whose element is $\underline{y} = (y_1, \dots, y_m)$ with $y_i \in \tilde{\Gamma}_{kb_i}^{\nu_i}$, $1 \leq i \leq m$. Then we define the following pseudo-orbit:

$$\mathfrak{C}_{\underline{y}} = \mathfrak{C}_{xx_1} \langle y_1, \dots, f^{kb_1} y_1 \rangle \mathfrak{C}_{x_{12}x}, \dots, \mathfrak{C}_{xx_{m_1}} \langle y_m, \dots, f^{kb_m} y_m \rangle \mathfrak{C}_{x_{m_2}x}, \mathfrak{C}_{xx_1}, \mathfrak{C}_{x_1x}, \dots, \mathfrak{C}_{xx_l}, \mathfrak{C}_{x_lx}.$$

It is clear that $\mathfrak{C}_{\underline{y}}$ is a δ -pseudo-orbit. Moreover, one notes that we can freely concatenate such $\mathfrak{C}_{\underline{y}}$ s to constitutes a δ -pseudo-orbit. So if we let $\Gamma^p = \Gamma \times \cdots \times \Gamma$ whose element is $\theta = (\theta_1, \dots, \theta_p)$ with $\theta_j \in \Gamma$, $1 \leq j \leq p$, we can define the following two δ -pseudo-orbits:

$$\tilde{\mathfrak{C}}_{\theta} = \mathfrak{C}_{\theta_1} \mathfrak{C}_{\theta_2} \cdots \mathfrak{C}_{\theta_p} \text{ and } \mathfrak{C}_{\theta} = \tilde{\mathfrak{C}}_{\theta} \tilde{\mathfrak{C}}_{\theta} \cdots \tilde{\mathfrak{C}}_{\theta} \cdots.$$

Hence \mathfrak{C}_{θ} can be ε -shadows by some point in X . Therefore, if we let

$$n = kb + \sum_{i=1}^m (l_{0i_1} + l_{i_2 0}) + \sum_{j=1}^l (l_{0j} + l_{j0}),$$

we can define the following nonempty set

$$\Gamma_{pn} := \{x \in X : x \varepsilon\text{-shadows some pseudo orbit } \mathfrak{C}_{\theta} \text{ with } \theta \in \Gamma^p\}.$$

By (4.14), $n \geq N$. To see that Γ_{pn} is an $(pn, \varepsilon^*/3)$ -separated set, consider any two points $x, x' \in \Gamma_{pn}$. Suppose $x \varepsilon$ -shadows \mathfrak{C}_{θ} and $x' \varepsilon$ -shadows $\mathfrak{C}_{\theta'}$ with $\theta_i \neq \theta'_i$ for some $1 \leq i \leq p$. Suppose that $\theta_i = (y_1, \dots, y_m)$ and $\theta'_i = (y'_1, \dots, y'_m)$. Then there exists a $1 \leq j \leq m$ such that $y_j \neq y'_j$. This implies that $d_{pn}(x, x') \geq \varepsilon_j - 2\varepsilon \geq \tilde{\varepsilon} - 2\varepsilon \geq \varepsilon^* - 2\varepsilon \geq \varepsilon^*/3$. Now let us prove that Γ_{pn} satisfies (a)(b)(c).

(a) For any $y \in \Gamma_{pn}$, $y \varepsilon$ -shadows some $\mathfrak{C}_{\theta} = \langle w_u \rangle_{u=0}^{\infty}$ with $\theta = (\theta_1, \dots, \theta_p)$ and $\theta_i = (\theta_{i,1}, \dots, \theta_{i,m})$, $\theta_{i,j} \in \tilde{\Gamma}_{kb_j}^{\nu_j}$, $1 \leq i \leq p$, $1 \leq j \leq m$. We have the following estimations:

1. $\rho\left(\sum_{j=1}^m \frac{b_j}{b} \nu_j, \mu\right) < a/4$.
2. $\rho\left(\sum_{j=1}^m \frac{b_j}{b} \mathcal{E}_{kb_j}(\theta_{i,j}), \sum_{j=1}^m \frac{b_j}{b} \nu_j\right) < a/4$ for each $1 \leq i \leq p$.
3. $\rho\left(\frac{1}{n} \sum_{t=(i-1)n}^{in-1} \delta_{w_t}, \sum_{j=1}^m \frac{b_j}{b} \mathcal{E}_{kb_j}(\theta_{i,j})\right) \leq \frac{n-kb}{kb+n-kb} \cdot 2 \leq \frac{2(m+l)L}{kb+2(m+l)L} \cdot 2 < a/4$ for each $1 \leq i \leq p$.
4. $\rho\left(\mathcal{E}_n(f^{(i-1)n}y), \frac{1}{n} \sum_{t=(i-1)n}^{in-1} \delta_{w_t}\right) < \varepsilon \leq a/4$ for each $1 \leq i \leq p$.

So we have $\rho(\mathcal{E}_n(f^{(i-1)n}y), \mu) < a/4 + a/4 + a/4 + a/4 = a$ for each $1 \leq i \leq p$. Therefore,

$$\rho(\mathcal{E}_{pn}(y), \mu) = \rho\left(\frac{1}{p} \sum_{i=1}^p \mathcal{E}_n(f^{(i-1)n}y), \mu\right) \leq \frac{1}{p} \sum_{i=1}^p \rho(\mathcal{E}_n(f^{(i-1)n}y), \mu) < a,$$

which implies that $y \in X_{pn, B(\mu, a)} \subseteq X_{pn, F_{\mu}}$. Consequently, $\Gamma_n \subseteq X_{pn, F_{\mu}}$. The fact that $\Gamma_{pn} \subseteq B(x, \varepsilon) \cap f^{-pn}B(x, \varepsilon)$ is obvious by the construction.

(b) Note that $|\Gamma_{pn}| = \prod_{i=1}^m |\tilde{\Gamma}_{kb_i}^{\nu_i}|^p \geq \prod_{i=1}^m \left(\frac{e^{kb_i(h_{\nu} - \frac{\eta}{3})}}{l^2}\right)^p$ by (4.17). Then by the affinity of the metric entropy and (4.16), we have

$$\frac{\log |\Gamma_{pn}|}{pn} \geq \frac{kb(h_{\mu} - \frac{2}{3}\eta) - 2m \log l}{n} \geq \frac{kb(h_{\mu} - \frac{2}{3}\eta) - 2m \log l}{kb + 2(m+l)L} > h_{\mu} - \eta.$$

(c) For any $y \in \Gamma_{pn}$, $y \varepsilon$ -shadows some $\mathfrak{C}_{\theta} \subseteq \Lambda$ with $\theta \in \Gamma^p$. Then $\cup_{i=0}^{pn-1} f^i \Gamma_{pn} \subseteq B(\Lambda, \varepsilon)$. Meanwhile, by the construction of \mathfrak{C}_{θ} , for any x_i , $1 \leq i \leq l$, there exists a point $z \in \cup_{i=0}^{pn-1} f^i \Gamma_{pn}$ which is ε close to x_i . However, $\{x_i\}_{i=1}^l$ is δ -dense in Λ . So $\cup_{i=0}^{pn-1} f^i \Gamma_{pn}$ is $(\delta + \varepsilon)$ -dense in Λ . Since $\delta + \varepsilon < 2\varepsilon$, we have $\Lambda \subseteq B(\cup_{i=0}^{pn-1} f^i \Gamma_{pn}, 2\varepsilon)$.

□

4.2 Star-entropy-dense and Strong-entropy-dense Properties

Definition 4.4. We say $M_{erg}(f, \Lambda)$ is star-entropy-dense in $M(f, \Lambda)$ if for any $\mu \in M(f, \Lambda)$, for any neighborhood G of μ in $M(\Lambda)$, and for any $\eta > 0$, there exists a $\nu \in M_{erg}(f, \Lambda)$ such that $h_\nu > h_\mu - \eta$ and $M(f, S_\nu) \subseteq G$.

Definition 4.5. We say $M_{erg}(f, \Lambda)$ is strong-entropy-dense in $M(f, \Lambda)$ if for any $\mu \in M(f, \Lambda)$, for any neighborhood G of μ in $M(\Lambda)$, and for any $\eta > 0$, there exists a closed f -invariant set Λ_μ such that $M(f, \Lambda_\mu) \subseteq G$ and $h_{top}(f, \Lambda_\mu) > h_\mu - \eta$.

Note 4.6. Of course, strong-entropy-dense \Rightarrow star-entropy-dense \Rightarrow entropy-dense. Moreover, for systems with the almost specification property, although Pfister and Sullivan used the entropy-dense property to achieve their goals, they had obtained the strong-entropy-dense properties by showing the following lemma.

Lemma 4.7. [57] Suppose that (X, f) has the almost specification property and that $\nu \in M(X)$ verifies the conclusions of lemma 2.19. Let $0 < h < h_\nu$. Then, there exists $\varepsilon > 0$ such that for any neighborhood G of ν , there exists a closed T -invariant subspace $Y \subseteq X$ and an $n_G \in \mathbb{N}$ with the following properties:

1. $\mathcal{E}_n(y) \in G$ whenever $n \geq n_G$ and $y \in Y$.
2. For all $l \in \mathbb{N}$ there exists a $(l \cdot n_G, \varepsilon)$ -separated subset of Y with cardinality greater than $\exp(l \cdot n_G \cdot h)$.

In particular, $h_{top}(f, Y) \geq h$.

Aside from the almost specification case, we also have the following result.

Proposition 4.8. Suppose (X, f) is topologically transitive and satisfies the shadowing property. Then (X, f) has the strong-entropy-dense property.

Proof. For any $\mu \in M(f, X)$, any neighborhood F_μ of μ and any $\eta > 0$, choose some $a > 0$ such that $\mathcal{B}(\mu, a) \subseteq F_\mu$. Fix an arbitrary $x \in X$. Since (X, f) is transitive and satisfies the shadowing property, by proposition A, there exists an $\varepsilon^* > 0$ such that for any $0 < \varepsilon \leq \varepsilon^*$, there exist an $n = n(F_\mu, \mu, \eta)$ and an $(n, \varepsilon^*/3)$ -separated set Γ_n so that

- (i) $\Gamma_n \subseteq X_{n, \mathcal{B}(\mu, a/4)} \cap B(x, \varepsilon) \cap f^{-n}B(x, \varepsilon)$;
- (ii) $\frac{\log |\Gamma_n|}{n} > h_\mu - \eta$;

Now let $\zeta = \min\{a/4, \varepsilon^*/9\}$. Then there exists a $\delta > 0$ such that any δ -pseudo-orbit can be ζ -shadowed. Then we set $\varepsilon = \min\{\varepsilon^*, \delta\}$. So we obtain an $(n, \varepsilon^*/3)$ -separated set Γ_n satisfying the property (i)(ii) above. For any $x \in \Gamma_n$, let $\mathfrak{C}_x = \langle x, f(x), \dots, f^n(x) \rangle$. By property (i) above, we can freely concatenate such \mathfrak{C}_x s to constitute an ε and thus a δ -pseudo-orbit. Let $\Sigma^+ = \prod_{i=1}^\infty \Gamma_n$ whose element is $\theta = (\theta_1, \theta_2, \dots)$ with each $\theta_i \in \Gamma_n$. Then we can define the following δ -pseudo-orbit:

$$\mathfrak{C}_\theta = \mathfrak{C}_{\theta_1}, \mathfrak{C}_{\theta_2}, \dots$$

By the shadowing property, there exists a $y \in X$ which ζ -shadows \mathfrak{C}_θ . So we can define the following nonempty set

$$\Delta := \{x \in X : x \text{ } \zeta \text{ shadows some } \mathfrak{C}_\theta, \theta \in \Sigma^+\}.$$

Of course, Δ is f^n -invariant and for each $x \in \Delta$,

$$\rho(\mathcal{E}_{in}(f^{jn}(x)), \mu) < \zeta + a/4 \leq a/2 \text{ for any } i, j \in \mathbb{N}. \quad (4.18)$$

Moreover, for any $i \in \mathbb{N}$, suppose x ζ -shadows \mathfrak{C}_θ and x' ζ -shadows $\mathfrak{C}_{\theta'}$ with $\theta_j \neq \theta'_j$ for some $1 \leq j \leq i$, then $d_{in}(x, y) > \varepsilon^*/3 - 2\zeta \geq \varepsilon^*/9$. This implies that for any $i \in \mathbb{N}$, there exists an $(i, \varepsilon^*/9)$ -separated set under f^n with cardinality $\geq |\Gamma_n|^i$ in Δ . So

$$h_{\text{top}}(f^n, \Delta) \geq \lim_{i \rightarrow \infty} \frac{\log |\Gamma_n|^i}{i} > n(h_\mu - \eta).$$

Let $\Lambda = \bigcup_{j=0}^{n-1} f^j \Delta$. One sees that Λ is f -invariant and

$$h_{\text{top}}(f, \Lambda) = \frac{1}{n} h_{\text{top}}(f^n, \Delta) > h_\mu - \eta.$$

Now choose $k \in \mathbb{N}$ large enough and define

$$\Xi := \left\{ x \in X : f^j(x) \in X_{kn, \overline{B(\mu, \frac{3}{4}a)}}, \forall j \in \mathbb{N} \right\}.$$

By definition, Ξ is f -invariant. To show Ξ is closed, suppose $x_l \rightarrow x$ as $l \rightarrow \infty$. Then for each $j \in \mathbb{N}$,

$$\rho(\mathcal{E}_{kn}(f^j(x_l)), \mu) \leq \frac{3}{4}a.$$

By the continuity of $\rho(\cdot, \cdot)$ and $\mathcal{E}_{kn}(f^j(\cdot))$, one sees that

$$\rho(\mathcal{E}_{kn}(f^j(x)), \mu) \leq \frac{3}{4}a,$$

which implies that Ξ is closed. Now let us prove that $\Lambda \subseteq \Xi$. Actually, for any $x \in \Xi$, $x = f^i(y)$ for some $0 \leq i \leq n-1$ and $y \in \Delta$. Then for any $j \in \mathbb{N}$, suppose $j + i = tn + r$, $0 \leq r \leq n-1$. By (4.18), one sees that $\rho(\mathcal{E}_{kn}(f^{tn}(y)), \mu) < a/2$. On the other hand, $\rho(\mathcal{E}_{kn}(f^{tn+r}(y)), \mathcal{E}_{kn}(f^{tn}(y))) \leq \frac{2r}{kn} \leq \frac{2}{k} \leq \frac{a}{4}$. So we get $\rho(\mathcal{E}_{kn}(f^{j+r}(y)), \mu) \leq \frac{3}{4}a$, i.e. $f^j(x) \in X_{kn, \overline{B(\mu, \frac{3}{4}a)}}$. Therefore, $\Lambda \subseteq \Xi$. Consequently,

$$h_{\text{top}}(f, \Xi) \geq h_{\text{top}}(f, \Lambda) > h_\mu - \eta.$$

Moreover, for any ergodic measure ν on Ξ , pick a generic point $y \in \Xi$ of ν . Then because $f^j(x) \in X_{kn, \overline{B(\mu, \frac{3}{4}a)}}$ for any $j \in \mathbb{N}$, one sees that

$$\rho(\nu, \mu) = \lim_{l \rightarrow \infty} \rho(\mathcal{E}_l(y), \mu) = \lim_{i \rightarrow \infty} \rho(\mathcal{E}_{ikn}(y), \mu) = \lim_{i \rightarrow \infty} \rho\left(\frac{1}{i} \sum_{j=0}^{i-1} \mathcal{E}_{kn}(f^{jn}(y)), \mu\right) \leq \frac{3}{4}a < a.$$

In particular, one has $M(f, \Xi) \subseteq F_\mu$. This Ξ serves as the Λ_μ we need. \square

For $k \in \mathbb{N}$, let $P_k(f) := \{x \in X : f^k x = x\}$.

Corollary A. *Let $\Lambda \subseteq X$ be compact f -invariant and internally chain transitive. Suppose (X, f) is topologically expanding (resp. topologically hyperbolic) and topologically transitive. Then for any $\eta > 0$, there exists an $\varepsilon^* = \varepsilon^*(\eta) > 0$ such that for any $\mu \in M(f, \Lambda)$ and its neighborhood F_μ , for any $x \in \Lambda$, for any $0 < \varepsilon \leq \varepsilon^*$, for any $N \in \mathbb{N}$, there exist an $n = n(F_\mu, \mu, \eta, \varepsilon) \geq N$ such that for any $p \in \mathbb{N}$, there exists an $(pn, \varepsilon^*/3)$ -separated set Γ_{pn} so that*

$$(1) \quad \Gamma_{pn} \subseteq X_{pn, F_\mu} \cap B(x, \varepsilon) \cap P_{pn}(f);$$

$$(2) \quad \frac{\log |\Gamma_{pn}|}{pn} > h_\mu - \eta;$$

$$(3) \quad d_H(\bigcup_{i=0}^{pn-1} f^i \Gamma_{pn}, \Lambda) < 2\varepsilon.$$

Proof. Since (X, f) is topologically transitive and has the shadowing property, (X, f) has the strong-entropy-dense property by the above proposition. Moreover, since (X, f) is expansive, (X, f) has the uniform separation property [58]. Therefore, by proposition 2.24, the $\tilde{\varepsilon}$ in the proof of lemma A can be chosen independent of μ and F_μ . Let $c > 0$ be the expansive constant, then modify the proof of lemma A by letting $\varepsilon^* = \min\{\frac{\tilde{\varepsilon}}{3}, \frac{a}{4}, \frac{c}{2}\}$. From the construction of the δ -pseudo-orbit \mathfrak{C}_θ , one sees that \mathfrak{C}_θ is a pn -periodic pseudo-orbit. So for any $w \in \Gamma_{pn}$, $d(f^{pn+t}w, f^tw) < \varepsilon + \varepsilon = 2\varepsilon \leq c$ for any $t \in \mathbb{Z}^+$. Therefore, $f^{pn}w = w$, which implies that $\Gamma_{pn} \subseteq P_{pn}(f)$. \square

4.3 Basic-entropy-dense and Strong-basic-entropy-dense Properties

Definition 4.9. We say (X, f) satisfies the basic-entropy-dense property if for any $\mu \in M(f, X)$, for any neighborhood G of μ , and for any $\eta > 0$, there exists a closed f -invariant set Λ_μ such that

- (1) Λ_μ is transitive and satisfies the shadowing property.
- (2) $h_{\text{top}}(f, \Lambda_\mu) > h_\mu - \eta$.
- (3) $M(f, \Lambda_\mu) \subseteq G$.

For any $m \in \mathbb{N}$ and $\{\nu_i\}_{i=1}^m \subseteq M(X)$, we write $\text{cov}\{\nu_i\}_{i=1}^m$ for the convex combination of $\{\nu_i\}_{i=1}^m$, namely,

$$\text{cov}\{\nu_i\}_{i=1}^m := \left\{ \sum_{i=1}^m t_i \nu_i : t_i \in [0, 1], 1 \leq i \leq m \text{ and } \sum_{i=1}^m t_i = 1 \right\}.$$

Definition 4.10. We say (X, f) satisfies the strong-basic-entropy-dense property if for any $K = \text{cov}\{\mu_i\}_{i=1}^m \subseteq M(f, X)$ and any $\eta, \zeta > 0$, there exist compact invariant subset $\Lambda_i \subseteq \Lambda \subsetneq X$, $1 \leq i \leq m$ such that

- 1. Λ is transitive and has the shadowing property.
- 2. For each $1 \leq i \leq m$, $h_{\text{top}}(f, \Lambda_i) > h_{\mu_i} - \eta$ and consequently, $h_{\text{top}}(f, \Lambda) > \sup\{h_\kappa : \kappa \in K\} - \eta$.
- 3. $d_H(K, M(f, \Lambda)) < \zeta$, $d_H(\mu_i, M(f, \Lambda_i)) < \zeta$.

Lemma B. Suppose (X, f) is topologically expanding and transitive. Then (X, f) satisfies the strong-basic-entropy-dense property.

Proof. Let $K = \text{cov}\{\nu_i\}_{i=1}^m \subseteq M(f, X)$ and $\eta, \zeta > 0$. By proposition 4.8, (X, f) has the strong-entropy-dense property, so there are infinitely many ergodic measures on X . This implies that $K \neq M(f, X)$ and thus $d_H(K, M(f, X)) > 0$. Fix an arbitrary $x \in X$. Then for each $1 \leq i \leq m$, by proposition A, there exist $\varepsilon_i^* > 0$ such that for any $0 < \varepsilon_i < \varepsilon_i^*$, there exists an $n_i \in \mathbb{N}$ such that for any $p \in \mathbb{N}$, there exists an $(pn_i, \varepsilon_i^*/3)$ -separated set $\Gamma_{pn_i}^{\nu_i}$ with

- (a) $\Gamma_{pn_i}^{\nu_i} \subseteq P_{n_i}(f) \cap X_{n_i, \mathcal{B}(\nu_i, \zeta/4)} \cap B(x, \varepsilon_i)$;
- (b) $\frac{\log |\Gamma_{pn_i}^{\nu_i}|}{pn_i} > h_{\nu_i} - \eta$.

Now let $\rho = \min\{\rho(\nu_i, \nu_j) : 1 \leq i < j \leq m\}$ and $\varepsilon^* = \min\{\varepsilon_i^* : 1 \leq i \leq m\}$. Set $\varepsilon = \min\{\zeta/4, \rho/3, \varepsilon^*/9\}$. Then there exists a $0 < \delta < \varepsilon$ such that any δ -pseudo-orbit can be ε -shadowed. Set $n = n_1 n_2 \cdots n_m$. Then for each $1 \leq i \leq m$, $P_{n_i}(f) \subseteq P_n(f)$ by definition and furthermore, we can obtain an $(n, \varepsilon^*/3)$ -separated set $\Gamma_n^{\nu_i}$ with

- (a) $\Gamma_n^{\nu_i} \subseteq P_n(f) \cap X_{n, \mathcal{B}(\nu_i, \zeta/4)} \cap B(x, \delta)$;

(b) $\frac{\log |\Gamma_n^{\nu_i}|}{n} > h_{\nu_i} - \eta$.

Now let $r_i = |\Gamma_n^{\nu_i}|$ and $r = \sum_{i=1}^m r_i$. Enumerate the elements of each $\Gamma_n^{\nu_i}$ by $\Gamma_n^{\nu_i} = \{p_1^i, \dots, p_{r_i}^i\}$. Let $\Sigma_{r_i}^+$ be the set of element $a = (a_0 a_1 a_2 \dots)$ such that $a_k \in \{p_1^i, \dots, p_{r_i}^i\}$, $k \in \mathbb{Z}^+$ and Σ_r^+ be the set of element $b = (b_0 b_1 b_2 \dots)$ such that $b_k \in \{p_1^1, \dots, p_{r_1}^1, \dots, p_1^m, \dots, p_{r_m}^m\}$, $k \in \mathbb{Z}^+$. For every θ in some $\Sigma_{r_i}^+$ or Σ_r^+ let

$$Y_\theta = \{z \in X : d(f^{in} z, \theta_i) \leq \varepsilon, i \in \mathbb{Z}^+\}.$$

Since $\Gamma_n^{\nu_i} \subseteq P_n(f) \cap B(x, \delta)$, for each θ in some $\Sigma_{r_i}^+$ or Σ_r^+ , we can freely concatenate the orbit segments $\mathfrak{C}_{\theta_i} = \langle \theta_i, \dots, f^n \theta_i \rangle$ to constitute a δ -pseudo-orbit $\mathfrak{C}_\theta = \mathfrak{C}_{\theta_0}, \mathfrak{C}_{\theta_1}, \mathfrak{C}_{\theta_2}, \dots$. So by the shadowing property, each Y_θ is nonempty and closed. Note that if $\theta \neq \theta'$, then there is $t \in \mathbb{Z}^+$ such that $\theta_t \neq \theta'_t$. But then there is $0 \leq e \leq n-1$ such that $d(f^e(p_{\theta_t}), f^e(p_{\theta'_t})) \geq \min\{\varepsilon^*/3, \rho\} - 2\varepsilon \geq \varepsilon$. This immediately implies that $Y_\theta \cap Y_{\theta'} = \emptyset$. So we can define the following disjoint union:

$$\Delta_i = \bigsqcup_{\theta \in \Sigma_{r_i}^+} Y_\theta \text{ and } \Delta = \bigsqcup_{\theta \in \Sigma_r^+} Y_\theta.$$

Note that $f^n(Y_\theta) \subseteq Y_{\sigma(\theta)}$. Then Δ_i , $1 \leq i \leq m$ and Δ are closed f^n -invariant sets. Therefore, if we define $\pi : \Delta \rightarrow \Sigma_r^+$ and $\pi_i : \Delta_i \rightarrow \Sigma_{r_i}^+$ as

$$\begin{aligned} \pi(x) &:= \theta \quad \text{for all } x \in Y_\theta \text{ with } \theta \in \Sigma_r^+, \\ \pi_i(x) &:= \theta' \quad \text{for all } x \in Y_{\theta'} \text{ with } \theta' \in \Sigma_{r_i}^+, \end{aligned}$$

then π and π_i are surjective by the shadowing property. Moreover, it is not hard to check that π and π_i are continuous. So Δ and Δ_i are closed. Meanwhile, (X, f) is expansive, so π, π_i are conjugations.

Let $\Lambda = \bigcup_{l=0}^{n-1} f^l \Delta$ and $\Lambda_i = \bigcup_{l=0}^{n-1} f^l \Delta_i$. Obviously, Λ and Λ_i are closed and f -invariant. Now let us prove that Λ and Λ_i satisfy the property 1-3.

1. Since $\pi : (\Delta, f^n) \rightarrow (\Sigma_r^+, \sigma)$ is a conjugation, the topologically mixing property and the shadowing property of (Σ_r^+, σ) yield the same properties of (Δ, f^n) . Therefore, proposition 4.1 ensures that (Λ, f) is topologically transitive and has the shadowing property.
2. One has $h_{\text{top}}(f, \Lambda_i) = \frac{1}{n} h_{\text{top}}(f^n, \Delta_i) = \frac{1}{n} h_{\text{top}}(\sigma, \Sigma_{r_i}^+) = \frac{\log |\Delta_i|}{n} > h_{\nu_i} - \eta$.
3. For any ergodic measure $\mu_i \in M_f(\Lambda_i)$, pick an arbitrary generic point z_i of μ_i . one has $\rho(\mu_i, \nu_i) < \zeta/4$. Then $\rho(\mathcal{E}_n(f^k z_i), \nu_i) < \varepsilon + \zeta/4 < \zeta$. In addition, we have $\mu_i = \lim_{l \rightarrow \infty} \mathcal{E}_l(z_i) = \lim_{k \rightarrow \infty} \mathcal{E}_{kn}(z_i)$. So we have $\rho(\mu_i, \nu_i) = \lim_{k \rightarrow \infty} \rho(\mathcal{E}_{kn}(z_i), \nu_i) < \zeta$. By the ergodic decomposition theorem, we obtain that $d_H(\nu_i, M_f(\Lambda_i)) < \zeta$. Now since K is convex and $\Lambda_i \subseteq \Lambda$, one gets that $K \subseteq \mathcal{B}(M(f, \Lambda), \zeta)$.

On the other hand, for any ergodic measure $\nu \in M(f, \Lambda)$, pick a generic point z of μ . Then z ε -shadows some δ -pseudo-orbit \mathfrak{C}_θ with $\theta \in \Sigma_r^+$. When $l \in \mathbb{N}$ is large enough, we have $\rho(\mathcal{E}_l(z), \nu) < \zeta/4$ and there exist integers $q_i, 1 \leq i \leq m$ such that

$$\rho\left(\mathcal{E}_l(z), \frac{\sum_{i=1}^m q_i \nu_i}{\sum_{i=1}^m q_i}\right) < \varepsilon + \frac{\zeta}{4} + \zeta/4 \leq \frac{3}{4}\zeta.$$

So $d_H(\nu, K) < \zeta$. By the ergodic decomposition theorem, $M(f, \Lambda) \subseteq \mathcal{B}(K, \zeta)$. As a result, $\Lambda \subsetneq X$. For otherwise, $d_H(K, M(f, \Lambda)) = d_H(K, M(f, X)) \geq \zeta$, a contradiction.

□

4.4 Minimal-entropy-dense Property

Definition 4.11. We say that (X, f) has the minimal-entropy-dense property if for any $\mu \in M(f, X)$ with $\int \varphi d\mu = a$ for some $a \in L_\varphi$, then for any $\eta > 0$, any $k \geq 2$ and there exists a $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$, there is a minimal subset $\overline{X} \subseteq X$ so that

1. there are exactly k ergodic measures $\{\nu_i\}_{i=1}^k$ on \overline{X} ;
2. $a - \gamma < \int \varphi d\nu_1 < a < \frac{1}{2}(a + L_2) < \int \varphi d\nu_2 < \cdots < \int \varphi d\nu_k$;
3. $h_{\nu_1} > h_\mu - 3\eta$.

Lemma C. Suppose (X, f) is topologically expanding (resp. topologically hyperbolic) and φ is a continuous function on X with $I_\varphi \neq \emptyset$. Then (X, f) has the minimal-entropy-dense property.

Proof. Select $\mu_{\min}, \mu_{\max} \in M(f, X)$ such that

$$\int \varphi d\mu_{\min} = \inf\{\int \varphi d\tau : \tau \in M(f, X)\} \text{ and } \int \varphi d\mu_{\max} = \sup\{\int \varphi d\tau : \tau \in M(f, X)\}.$$

For any $\mu \in M(f, X)$ with $\int \varphi d\mu = a$ for some $a \in L_\varphi$, any $\eta > 0$ and any $k \geq 2$, set $\gamma_0 = \min\{a - L_1, \frac{L_2 - a}{k + \frac{1}{2}}, \frac{2\eta}{(a - L_1)h_\mu}\}$. For any $0 < \gamma < \gamma_0$, let $\theta = \frac{a - L_1 - \gamma/2}{a - L_1}$. Then $\mu_1 := \theta\mu + (1 - \theta)\mu_{\min}$ satisfies

$$\int \varphi d\mu_1 = a - \gamma/2 \text{ and } h_{\mu_1} \geq \theta h_\mu > h_\mu - \eta. \quad (4.19)$$

Since $k\gamma < L_2 - a - \gamma/2$, there exist $1 > \theta_2 > \cdots > \theta_k > 0$ such that $\mu_i = \theta_i\mu + (1 - \theta_i)\mu_{\max}$ satisfies

$$\frac{1}{2}(a + L_2) + \gamma/2 < \int \varphi d\mu_2 < \cdots < \int \varphi d\mu_k. \quad (4.20)$$

and

$$\int \varphi d\mu_i - \int \varphi d\mu_{i-1} > \gamma, i = 2, \dots, k. \quad (4.21)$$

Now let $K := \text{cov}\{\mu_i\}_{i=1}^k$. Choose $\zeta > 0$ such that $\rho_X(\mu, \nu) < \zeta \Rightarrow |\int \varphi d\mu - \int \varphi d\nu| < \gamma/2$. By lemma B, we obtain an closed f^k -invariant set Δ such that $\pi : (\Delta, f^n) \rightarrow (\Sigma_r^+, \sigma)$ and $\pi_i : (\Delta_i, f^n) \rightarrow (\Sigma_r^+, \sigma)$ are conjugations. Moreover, if we let $\Lambda = \bigcup_{l=0}^{n-1} f^l \Delta$ and $\Lambda_i = \bigcup_{l=0}^{n-1} f^l \Delta_i$, then Λ, Λ_i are closed f -invariant and satisfy

1. Λ is transitive and has the shadowing property.
2. For each $1 \leq i \leq m$, $h_{\text{top}}(f, \Lambda_i) > h_{\mu_i} - \eta$ and consequently, $h_{\text{top}}(f, \Lambda) > \sup\{h_\kappa : \kappa \in K\} - \eta$.
3. $d_H(K, M(f, \Lambda)) < \zeta$, $d_H(\mu_i, M(f, \Lambda_i)) < \zeta$.

By the variational principle, we choose k ergodic measures $\omega_i \in M(f, \Lambda_i), i = 1, \dots, k$ such that

$$h_{\omega_i} > h_{\text{top}}(f, \Lambda_i) - \eta > h_{\mu_i} - 2\eta \text{ and } \rho_X(\omega_i, \mu_i) < \zeta/2. \quad (4.22)$$

By proposition 4.2, there is an isometric isomorphism j between $M_{\text{erg}}(f^n, \Delta)$ and $M_{\text{erg}}(M, \Lambda)$. So for each ω_i , we obtain an $\tilde{\omega}_i = j^{-1}(\omega_i) \in M_{\text{erg}}(f^n, \Delta)$. Moreover, each $\tilde{\omega}_i$ is pushed forward by π to an ergodic measure $\omega^i = \pi_* \tilde{\omega}_i \in \mathfrak{M}_\sigma(\Sigma_r^+)$. Then we use proposition 4.3 to find a minimal subsystem $\overline{M} \subseteq \Sigma_r^+$ supporting exactly k ergodic measures $\{\nu^j\}_{j=1}^k$ such that

1. $\rho(\nu^j, \omega^j) < \zeta/2$ for each $j = 1, \dots, k$.

2 $h_{\nu^j} > h_{\omega^j} - \eta, j = 1, \dots, k$.

Let $\overline{X} = \pi^{-1}(\overline{M})$. It is left to show that \overline{X} is the one we need. Indeed, since π is a conjugation, (\overline{X}, f^n) supports exactly k ergodic measures $\tilde{\nu}_i := (\pi^{-1})_* \nu^i \subseteq M(f^n, \overline{X})$. Note that \overline{X} is f^n -minimal and $AP(f^n) = AP(f)$. So \overline{X} is f -minimal. Now let $\nu_i = j(\tilde{\nu}_i)$. Then

$$\rho_{\Delta}(\tilde{\omega}_i, \tilde{\nu}_i) = \rho_{\Sigma_r^+}(\omega^i, \nu^i) < \zeta/2.$$

Moreover, from [75] we see

$$h_{\tilde{\nu}_i}(f^n) = h_{\nu^i}(\sigma) > h_{\omega^i}(\sigma) - \eta = h_{\tilde{\omega}_i}(f^n) - \eta. \quad (4.23)$$

Furthermore, by proposition 4.2, we see

$$\rho_X(\omega_i, \nu_i) = \rho_{\Delta_i}(\tilde{\omega}_i, \tilde{\nu}_i) < \zeta/2.$$

This along with (4.22) shows that $\rho_X(\mu_i, \nu_i) < \zeta$ for each $i = 1, \dots, k$. By the selection of ζ , we see that

$$\left| \int \varphi d\mu_i - \int \varphi d\nu_i \right| < \gamma/2, i = 1, \dots, k.$$

This along with (4.19), (4.20) and (4.21) shows that

$$a - \gamma < \int \varphi d\nu_1 < a < \frac{1}{2}(a + L_2) < \int \varphi d\nu_2 < \dots < \int \varphi d\nu_k.$$

In addition, proposition 4.2 gives that

$$nh_{\nu_i}(f) = h_{\tilde{\nu}_i}(f^n) \text{ and } h_{\tilde{\omega}_i}(f^n) = nh_{\omega_i}(f).$$

This along with (4.19), (4.22) and (4.23) shows that

$$h_{\nu_1} > h_{\mu} - 3\eta.$$

The proof is completed. □

Corollary 4.12. Suppose (X, f) is topologically expanding (resp. topologically hyperbolic) and transitive. If φ is a continuous function on X with $I_{\varphi} \neq \emptyset$, then for any $\eta > 0$ and any $\nu \in M(f, X)$, there is a minimal subset $\overline{X} \subseteq X$ such that

1. there are exactly two ergodic measures $\{\nu_i\}_{i=1}^2$ on \overline{X} ;
2. $\int \varphi d\nu_1 < \int \varphi d\nu_2$;
3. $h_{\nu_1} > h_{\nu} - \eta$.

5 Saturated Set and Saturated Properties

In this section, we consider a topological dynamical system (X, f) . For $x \in X$, we denote the set of limit-point points of $\{\mathcal{E}_n(x)\}$ by $V_f(x)$. As is known, $V_f(X)$ is a non-empty compact connected subset of $M(f, X)$ [58]. So for any non-empty compact connected subset K of $M(f, X)$, it is logical to define the following set

$$G_K := \{x \in X \mid V_f(x) = K\}.$$

We call G_K the saturated set of K . Particularly, if $K = \{\mu\}$ for some ergodic measure μ , then G_{μ} is just the generic points of μ . Saturated sets are studied by Pfister and Sullivan in [58]. Furthermore, we define $G_K^T = \{x \in \text{Tran}(f) : V_f(x) = K\}$ and call G_K^T the transitively-saturated set of K .

Definition 5.1. We say that the system f has *saturated* property or f is *saturated*, if for any compact connected nonempty set $K \subseteq M(f, X)$,

$$h_{top}(f, G_K) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.24)$$

We say that the system f has *locally-saturated* property or f is *locally-saturated*, if for any open set $U \subseteq X$ and any compact connected nonempty set $K \subseteq M(f, X)$,

$$h_{top}(f, G_K \cap U) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.25)$$

In parallel, one can define (locally) *transitively-saturated*, just replacing G_K by G_K^T in (5.24) (resp., (5.25)). On the other hand, (locally, transitively-)single-saturated means (5.24) (resp., (5.25)) holds when K is a singleton.

When the dynamical system f satisfies g -almost product and uniform separation property, Pfister and Sullivan proved in [58] that f is saturated and Huang, Tian and Wang proved in [38] that f is transitively-saturated.

5.1 Star and Locally-star Saturated Properties

Let $\Lambda \subseteq X$ be a closed invariant subset and K is a non-empty compact connected subset of $M(f, \Lambda)$. Define

$$G_K^\Lambda := G_K \cap \{x \in X \mid \omega_f(x) = \Lambda\}.$$

Lemma 5.2. For (X, f) , let $\Lambda \subsetneq X$ be closed f -invariant and $K \subseteq M(f, \Lambda)$ be a nonempty compact connected set.

(1) Suppose $\bigcap_{\mu \in K} S_\mu = C_\Lambda$. Then

$$G_K^\Lambda \subseteq \{x \in X : X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

(2) Suppose $\bigcap_{\mu \in K} S_\mu = \overline{\bigcup_{\nu \in K} S_\nu} \subsetneq C_\Lambda$. Then

$$G_K^\Lambda \subseteq \{x \in X : X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

(3) Suppose $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} = C_\Lambda$. Then

$$G_K^\Lambda \subseteq \{x \in X : X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

(4) Suppose $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} \subsetneq C_\Lambda$. Then

$$G_K^\Lambda \subseteq \{x \in X : X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Proof. For any $x \in G_K^\Lambda$, $\omega_f(x) = \Lambda$ by definition. So by corollary 3.4,

$$X_{B^*}(x) = C_x = \overline{\bigcup_{\mu \in M(f, \omega_f(x))} S_\mu} = \overline{\bigcup_{\mu \in M(f, \Lambda)} S_\mu} = C_\Lambda.$$

Consequently, one uses 1.1 and obtains that

$$X_{\underline{d}}(x) \subseteq X_{\overline{d}}(x) \subseteq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda. \quad (5.26)$$

Moreover, since $V_f(x) = K$ by definition, proposition 3.3 gives that

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu \text{ and } X_{\overline{d}}(x) = \overline{\bigcup_{\mu \in V_f(x)} S_\mu} = \overline{\bigcup_{\nu \in K} S_\nu}. \quad (5.27)$$

Therefore, a convenient use of (5.26) and (5.27) yields (1)-(4). \square

Definition 5.3. We say that Λ is star-saturated, if for any non-empty connected compact set $K \subseteq M(f, \Lambda)$, one has

$$h_{top}(f, G_K^\Lambda) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.28)$$

We say that Λ is locally-star-saturated, if for any open set $U \subseteq X$ and any compact connected nonempty set $K \subseteq M(f, \Lambda)$,

$$h_{top}(f, G_K^\Lambda \cap U) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.29)$$

We say that the system f has *star-saturated* property or f is *star-saturated*, if for any internally chain transitive closed invariant subset $\Lambda \subseteq X$, Λ is star-saturated. We say that the system f has *locally-saturated* property or f is *locally-saturated*, if for any internally chain transitive closed invariant subset $\Lambda \subseteq X$, Λ is locally star-saturated.

Note that X being star-saturated $\Leftrightarrow f$ is transitively-saturated, and for a closed invariant subset $\Lambda \subsetneq X$, $f|_\Lambda$ is transitively-saturated $\Rightarrow \Lambda$ being star-saturated. However, remark that Λ being star-saturated does not imply $f|_\Lambda$ is transitively-saturated, since some points of G_K^Λ may not lie in Λ .

Proposition A. *Suppose (X, f) is topologically transitive and topologically expanding (resp., a transitive and topologically hyperbolic homeomorphism). Then f is locally-star-saturated.*

Proof. We need to show that for any internally chain transitive closed invariant subset $\Lambda \subseteq X$, any open set $U \subseteq X$ and any nonempty compact connected subset $K \subseteq M(f, \Lambda)$, one has

$$h_{top}(f, G_K^\Lambda \cap U) = \inf\{h_\mu(f) \mid \mu \in K\}.$$

Since $G_K^\Lambda \subseteq G_K$ by definition and $h_{top}(G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}$ by lemma 2.1, one has

$$h_{top}(G_K^\Lambda \cap U) \leq \inf\{h_\mu(f) \mid \mu \in K\}.$$

So it remains to show that

$$h_{top}(G_K^\Lambda \cap U) \geq \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.30)$$

To begin with, we recall the following conclusion which was used in the proof of [25, Proposition 21.14]. A proof can be found, for example, in [27].

Lemma 5.4. For any non-empty compact connected set $K \subseteq \mathcal{M}_f(X)$, there exists a sequence of open balls B_n in $\mathcal{M}_f(X)$ with radius ζ_n such that the following holds:

- (a) $B_n \cap B_{n+1} \cap K \neq \emptyset$;
- (b) $\bigcap_{N=1}^\infty \bigcup_{n \geq N} B_n = K$;
- (c) $\lim_{n \rightarrow \infty} \zeta_n = 0$.

This allows us to pick an $\alpha_n \in B_n \cap B_{n+1} \cap K$ for each n . Then

$$d(\alpha_n, \alpha_{n+1}) < \zeta_{n+1} \text{ for each } n. \quad (5.31)$$

Moreover, $\{\alpha_n\}_{n \geq 1}$ are dense in K due to property (b) and (c) there.

Fix an arbitrary $\eta > 0$ and an $x \in U$. Let $F_k = \mathcal{B}(\alpha_k, \zeta_k)$ for each $k \in \mathbb{N}$. Since (X, f) is topologically transitive and topologically expanding, by lemma A, there is an $\varepsilon^* > 0$ such that for any $0 < \varepsilon_k \leq \varepsilon^*$ and any $N \in \mathbb{N}$, there is an $n_k \geq N$ and an $(n_k, \varepsilon^*/3)$ -separated set $\Gamma_{n_k}^{\alpha_k}$ so that

$$(a) \quad \Gamma_{n_k}^{\alpha_k} \subseteq X_{n_k, F_k} \cap B(x, \varepsilon_k) \cap P_{n_k}(f);$$

$$(b) \quad \frac{\log |\Gamma_{n_k}^{\alpha_k}|}{n_k} > h_{\alpha_k} - \eta;$$

$$(c) \quad d_H(\cup_{i=0}^{n_k-1} f^i \Gamma_{n_k}^{\alpha_k}, \Lambda) < 2\varepsilon_k.$$

Now let $d = \text{dist}(x, X \setminus U)$ and c be the expansive constant. Set

$$\varepsilon := \min\{\varepsilon^*/9, c/2, d/2\}.$$

By the shadowing property, there is a $0 < \delta < \varepsilon$ such that any δ -pseudo-orbit can be ε -shadowed. Then we set

$$\varepsilon_k = 2^{-k}\delta, k \in \mathbb{N}.$$

Now choose a strictly increasing positive integer sequence N_k such that

$$\lim_{k \rightarrow \infty} \frac{n_k}{\sum_{j=1}^{k-1} N_j n_j} = 0 \quad (5.32)$$

and

$$\lim_{k \rightarrow \infty} \frac{N_k n_k}{\sum_{j=1}^{k-1} N_j n_j} = +\infty. \quad (5.33)$$

Moreover, we define the stretched sequences $\{n'_j\}$, $\{\epsilon'_j\}$ and $\{\zeta'_j\}$, by setting for

$$j = N_1 + \cdots + N_{k-1} + q \text{ with } 1 \leq q \leq N_k,$$

$$n'_j := n_k \quad \Gamma'_j := \Gamma_{n_k}^{\alpha_k}.$$

Let $\mathcal{W}_k := \prod_{i=1}^k \Gamma'_i$ whose element is $\underline{w} = (w_1 \cdots w_k)$ with $w_i \in \Gamma'_i$ for $1 \leq i \leq k$. For each $w_j \in \Gamma'_j$, denote

$$\mathfrak{C}_{w_j} = \langle w_j, \dots, f^{n'_j} w_j \rangle.$$

Due to the property (a) of $\Gamma_{n_k}^{\alpha_k}$ and the fact that $\varepsilon_i + \varepsilon_j \leq \delta, i, j \in \mathbb{N}$, we can freely concatenate \mathfrak{C}_k s to constitute a δ -pseudo-orbit. In particular, for each $k \in \mathbb{N}$ and $\underline{w} \in \mathcal{W}_k$, there corresponds a δ -chain

$$O_w := \mathfrak{C}_{w_1} \mathfrak{C}_{w_2} \cdots \mathfrak{C}_{w_k}$$

which can be ε -shadowed. So the following defined set

$$G(\underline{w}) := \bigcap_{j=1}^k f^{-M_{j-1}} \overline{B_{n'_j}(w_j, \varepsilon)}$$

is nonempty and closed. Furthermore, we define

$$G_k := \bigcap_{j=1}^k \left(\bigcup_{w_j \in \Gamma'_j} f^{-M_{j-1}} \overline{B_{n'_j}(w_j, \varepsilon)} \right) \text{ with } M_j := \sum_{l=1}^j n'_l.$$

It is clear that G_k is nonempty and closed. Moreover, we have

Lemma 5.5. $G_k = \bigsqcup_{\underline{w} \in \mathcal{W}_k} G(\underline{w})$. Here \bigsqcup denotes the disjoint union.

Proof. Of course, $G_k = \bigcup_{\underline{w} \in \mathcal{W}_k} G(\underline{w})$. Moreover, if there exist $\underline{u}, \underline{v} \in \mathcal{W}_k, \underline{u} \neq \underline{v}$ such that $G(\underline{u}) \cap G(\underline{v}) \neq \emptyset$, then there exists an $1 \leq i \leq k$ such that $u_i \neq v_i$. This implies that $d_{n'_i}(u_i, v_i) \geq \varepsilon^*/3 \geq 3\varepsilon$. However, by choosing an $\underline{x} \in G(\underline{u}) \cap G(\underline{v})$, we see that $d_{n'_i}(x_i, u_i) < \varepsilon$ and $d_{n'_i}(x_i, v_i) < \varepsilon$, which implies that $d_{n'_i}(u_i, v_i) < 2\varepsilon$, a contradiction. So $G(\underline{w})$ are pairwise disjoint. \square

Furthermore, let $\mathcal{W} := \prod_{j=1}^{+\infty} \Gamma'_j$ whose element is $w = (w_1 w_2 \cdots)$ with $w_j \in \Gamma'_j$, $j \geq 1$. For each $w \in \mathcal{W}$, there corresponds a δ -pseudo-orbit

$$O_w := \mathfrak{C}_{w_1} \mathfrak{C}_{w_2} \cdots \mathfrak{C}_{w_k} \cdots$$

which can be ε -shadowed. So the following defined set

$$G(w) := \bigcap_{j=1}^{\infty} f^{-M_{j-1}} \overline{B_{n'_j}(w_j, \varepsilon)}$$

is nonempty and closed.

Lemma 5.6. For any $w \in \mathcal{W}$, O_w is a δ -limit-pseudo-orbit and $G(w)$ contains exactly one point. Moreover, this point ε -limit-shadows O_w .

Proof. The construction of O_w and the fact that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ shows that O_w is a δ -limit-pseudo-orbit. Therefore, O_w is ε -limit-shadowed by a point $y \in X$ due to lemma 2.14. Since (X, f) is positively expansive with expansive constant c and $\varepsilon \leq c/2$, y is uniquely defined. Moreover, if we denote $O_w = \{z_n\}_{n=0}^{\infty}$, then

$$d(f^n y, z_n) \leq \varepsilon \leq c/2, n \in \mathbb{Z}^+$$

and for any $x \in G(w)$,

$$d(f^n x, z_n) \leq \varepsilon \leq c/2, n \in \mathbb{Z}^+.$$

Therefore, $d(f^n x, f^n y) \leq c$ for any $n \geq \mathbb{Z}^+$, which implies that $x = y$. Consequently, $G(w) = \{y\}$, which concludes our proof. \square

Let

$$G := \bigcap_{k \geq 1} G_k.$$

It is clear that G are nonempty and closed. In addition, a similar discussion to lemma 5.5 shows that

$$G = \bigsqcup_{w \in \mathcal{W}} G(w). \quad (5.34)$$

Therefore, for any $x \in G$, there is a $w = w(x) \in \mathcal{W}$ such that $x \in G(w)$. Moreover, by lemma 5.6, $G(w) = \{x\}$. Consequently, there is a unique correspondence between G and \mathcal{W} .

Now we show that

$$G \subseteq G_K^\Lambda \cap U. \quad (5.35)$$

Indeed, $G \subseteq U$ is clear by construction. To show that for any $x \in G$, $\omega(x) = \Lambda$, suppose x ε -limit-shadows some $O_w = \{y_i\}_{i=0}^{+\infty}$ with $w \in \mathcal{W}$. Due to property (c) of $\Gamma_{n_k}^\alpha$, one sees that $\omega(\{y_i\}_{i=0}^{+\infty}) = \Lambda$. Meanwhile, lemma 2.15 ensures that $\omega(x) = \omega(\{y_i\}_{i=0}^{+\infty}) = \Lambda$. Then we are left to prove

Lemma 5.7. For any $x \in G$, $V_f(x) = G_K$.

Proof. Since $\{\alpha_k\}_{k=1}^{\infty}$ is dense in K , it suffices to prove that $\{\mathcal{E}_n(x)\}_{n=1}^{\infty}$ has the same limit-points as $\{\alpha_k\}_{k=1}^{\infty}$. To this end, set $S_k := \sum_{l=1}^k n_l N_l$ with $S_0 := 0$ and define the stretched sequence $\{\alpha'_m\}$ by

$$\alpha'_m := \alpha_k \text{ if } S_{k-1} < m \leq S_k.$$

The sequence $\{\alpha'_m\}$ has the same limit-point set as the sequences $\{\alpha_k\}$.

Now suppose x ε -limit-shadows some $O_w = \{y_n\}_{n=0}^{+\infty}$ with $w \in \mathcal{W}$, then $\lim_{n \rightarrow \infty} d(f^n x, y_n) = 0$. Let $\beta_n = \sum_{i=0}^{n-1} \delta_{y_i}$. Using the metric (2.4), we have $\lim_{n \rightarrow \infty} \rho(\mathcal{E}_n(x), \beta_n) = 0$. Hence, $\mathcal{E}_n(x)$ shares the same limit-points as the sequence β_n . So we are left to prove that

$$\lim_{n \rightarrow \infty} \rho(\alpha'_n, \beta_n) = 0.$$

In fact, by lemma 2.2 and (5.32), we only need to prove that

$$\lim_{k \rightarrow \infty} \rho(\alpha'_{M_k}, \beta_{M_k}) = 0.$$

Indeed, lemma 2.2 and (5.33) indicates that

$$\lim_{n \rightarrow \infty} \rho(\beta_{S_n}, \alpha_n) = 0. \quad (5.36)$$

Now for each $k \in \mathbb{N}$, there are $j \in \mathbb{N}$ and $1 \leq r \leq N_{j+1}$ such that $M_k = S_j + rn_{j+1}$. Then

$$\alpha'_{M_k} = \alpha_{j+1} \text{ and } \beta_{M_k} = \frac{S_j \beta_{S_j} + \sum_{p=1}^r n_{j+1} \mathcal{E}_{n_{j+1}}(w_{N_1+\dots+N_k+p})}{M_k}.$$

However, since $w_t \in \Gamma'_t$ for each $t \in \mathbb{N}$,

$$\rho(\mathcal{E}_{n_{j+1}}(w_{N_1+\dots+N_k+p}), \alpha_{j+1}) < \zeta_{j+1} \text{ for } 1 \leq p \leq r$$

Therefore, one has

$$\begin{aligned} \rho(\beta_{M_k}, \alpha_{j+1}) &\leq \frac{S_j}{M_k} [\rho(\beta_{S_j}, \alpha_j) + \rho(\alpha_j, \alpha_{j+1})] + \frac{n_{j+1}}{M_k} \sum_{p=1}^r \rho(\mathcal{E}_{n_{j+1}}(w_{N_1+\dots+N_k+p}), \alpha_{j+1}) \\ &\leq \rho(\beta_{S_j}, \alpha_j) + \zeta_{j+1} + \zeta_{j+1}. \end{aligned}$$

The proof will be completed if one notices (5.36) and that $\lim_{i \rightarrow \infty} \zeta_i = 0$. □

Let $\tilde{h} = \inf\{h_\mu : \mu \in K\}$ and $h = \tilde{h} - 2\eta$. We shall prove that

$$h_{\text{top}}(f, G) \geq h. \quad (5.37)$$

Recall the definition in (2.8). Since $C(G; s, \sigma, f)$ is non-decreasing as σ decreases, it is enough to prove that there exists $\tilde{\sigma} > 0$ such that

$$C(G; h, \tilde{\sigma}, f) \geq 1. \quad (5.38)$$

In fact, we will prove that (5.38) holds for $\tilde{\sigma} = \varepsilon/2$.

Due to (5.32), there exists a $q \in \mathbb{N}$ such that

$$\frac{M_r}{M_{r+1}} \geq \frac{\tilde{h} - 2\eta}{\tilde{h} - \eta} \text{ for any } r \geq q. \quad (5.39)$$

Now let $N = M_q$, it suffices to prove that

$$C(G; h, n, \varepsilon/2, f) \geq 1 \text{ for any } n \geq N. \quad (5.40)$$

Alternatively, one should show that for any $\mathcal{C} \in \mathcal{G}_n(G, \frac{\varepsilon}{2})$ with $n \geq N$,

$$\sum_{B_u(z, \frac{\varepsilon}{2}) \in \mathcal{C}} e^{-hu} \geq 1. \quad (5.41)$$

Moreover, we can assume that if $B_u(z, \frac{\varepsilon}{2}) \in \mathcal{C}$ then $B_u(z, \frac{\varepsilon}{2}) \cap G \neq \emptyset$, since otherwise we may remove this set from \mathcal{C} and it still remains as cover of G . In addition, for each $\mathcal{C} \in \mathcal{G}_n(G, \frac{\varepsilon}{2})$, we define a cover \mathcal{C}' in which each ball $B_u(z, \frac{\varepsilon}{2})$ is replaced by $B_{S_m}(z, \frac{\varepsilon}{2})$, $S_m \leq u < S_{m+1}$, $m \geq q$. Then

$$\sum_{B_u(x, \frac{\varepsilon}{2}) \in \mathcal{C}} e^{-hu} \geq \sum_{B_{M_m}(x, \frac{\varepsilon}{2}) \in \mathcal{C}'} e^{-hM_{m+1}}. \quad (5.42)$$

Consider a specific \mathcal{C}' and let c be the largest number of p for which there exists $B_{M_p}(x, \varepsilon) \in \mathcal{C}'$. Define

$$\mathcal{V}_c := \bigcup_{l=1}^c \mathcal{W}_l. \quad (5.43)$$

For $1 \leq j \leq k$, we say $\underline{w} \in \mathcal{W}_j$ is a prefix of $\underline{v} \in \mathcal{W}_k$ if the first j entries of \underline{w} coincides with \underline{v} , namely $w_i = v_i$ for $1 \leq i \leq j$. Note that each word of \mathcal{W}_m is a prefix of exactly $|\mathcal{W}_c|/|\mathcal{W}_m|$ words of \mathcal{W}_c .

Lemma 5.8. If $\mathcal{V} \subset \mathcal{V}_c$ contains a prefix of each word of \mathcal{W}_c , then

$$\sum_{m=1}^c |\mathcal{V} \cap \mathcal{W}_m| \frac{|\mathcal{W}_c|}{|\mathcal{W}_m|} \geq |\mathcal{W}_c|. \quad (5.44)$$

Consequently,

$$\sum_{m=1}^c |\mathcal{V} \cap \mathcal{W}_m| / |\mathcal{W}_m| \geq 1. \quad (5.45)$$

Proof. For any $\underline{w} \in \mathcal{W}_c$, there is an $1 \leq m \leq c$ such that $(w_1, \dots, w_m) \in (\mathcal{V} \cap \mathcal{W}_m)$. However, for each $\underline{v} \in \mathcal{W}_m$, the number of $\underline{w} \in \mathcal{W}_c$ such that $(w_1, \dots, w_m) = \underline{v}$ does not exceed $|\mathcal{W}_c|/|\mathcal{W}_m|$. This proves (5.44) and hence (5.45). \square

By lemma 5.5, each $x \in B_{M_m}(z, \frac{\varepsilon}{2}) \cap G$ corresponds uniquely to a point in \mathcal{W}_m . So \mathcal{C}' corresponds uniquely to a subset $\mathcal{V} \subset \mathcal{V}_c$ such that \mathcal{V} contains a prefix of each word of \mathcal{V}_c . This combining with (5.45) gives that

$$\sum_{B_{M_m}(x, \varepsilon) \in \mathcal{C}'} \frac{1}{|\mathcal{W}_m|} \geq 1. \quad (5.46)$$

Since

$$|\mathcal{W}_m| = 2^{\prod_{j=1}^m |\Gamma_j'|} \geq 2^{\prod_{j=1}^m n_j'(h_{\alpha_j'} - \eta)} \geq 2^{M_m(\tilde{h} - \eta)},$$

one has

$$\sum_{B_{M_m}(Z, \frac{\varepsilon}{2}) \in \mathcal{C}'} 2^{-M_m(\tilde{h} - \eta)} \geq 1.$$

Moreover, since $m \geq n \geq N$, (5.39) produces that

$$\sum_{B_{M_m}(Z, \frac{\varepsilon}{2}) \in \mathcal{C}'} 2^{-M_{m+1}(\tilde{h} - 2\eta)} \geq 1.$$

Note that $h = \tilde{h} - 2\eta$. Using (5.42), one gets that

$$\sum_{B_m(Z, \frac{\varepsilon}{2}) \in \mathcal{C}} 2^{-mh} \geq 1 \Rightarrow C(G; h, n, \varepsilon, f) \geq 1.$$

Finally, (5.35), (5.37) and the arbitrariness of η lead to (5.30). The proof is completed. \square

Let $\omega_f = \{A \subseteq X : \exists x \in X \text{ with } \omega_f(x) = A\}$ and denote the collection of internally chain transitive sets by $ICT(f)$. For any open $U \subseteq X$, define

$$\omega_f^U := \{A \subseteq X : \text{there exists } x \in X \text{ such that } \omega_f(x) = A\}.$$

Corollary 5.9. Suppose (X, f) is topologically transitive and topologically expanding (resp., a transitive and topologically hyperbolic homeomorphism). Then for any nonempty open $U \subseteq X$, $ICT(f) = \omega_f = \omega_f^U$.

Proof. According to [37, Lemma 2.1], we know $\omega_f \subseteq ICT(f)$. On the other hand, for any internally chain transitive set $\Lambda \subset X$, using the proposition above, we see in particular that there exists an $x \in U$ with $\omega_f(x) = \Lambda$, which implies that $\omega_f^U \supseteq ICT(f)$. \square

Remark 5.10. Meddaugh and Raines [47] establish that, for maps f with shadowing, $\overline{\omega_f} = ICT(f)$. Here we give a sufficient condition to ensure $\omega_f = ICT(f)$. Recently we learned that [34] Good and Meddaugh provide a both sufficient and necessary condition for $\omega_f(x) = ICT(f)$: $\omega_f(x) = ICT(f)$ if and only if f satisfies Pilyugin's notion of orbital limit shadowing.

5.2 Nonrecurrently-star-saturated Property

Define

$$G_K^N := G_K \cap NR(f) \text{ and } G_K^{\Lambda, N} := G_K^\Lambda \cap G_K^N$$

Definition 5.11. We say that Λ is nonrecurrently-star-saturated, if for any non-empty connected compact set $K \subseteq M(f, \Lambda)$, one has

$$h_{top}(f, G_K^{\Lambda, N}) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.47)$$

Proposition 5.12. For (X, f) , let $\Lambda \subsetneq X$ be a closed f -invariant subset. Suppose $(\Lambda, f|_\Lambda)$ is locally-star-saturated. Then Λ is nonrecurrently-star-saturated.

Proof. For any non-empty connected compact set $K \subseteq M(f, \Lambda)$, one has by lemma 2.1

$$h_{top}(f, G_K^{\Lambda, N}) \leq h_{top}(f, G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}.$$

It is left to show that

$$h_{top}(f, G_K^{\Lambda, N}) \geq \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.48)$$

Indeed, choose a nonempty open set $U \subset X \setminus \Lambda$, by the locally-star-saturated property, one has

$$h_{top}(f, G_K^\Lambda \cap U) = \inf\{h_\mu(f) \mid \mu \in K\}. \quad (5.49)$$

However, for any $x \in G_K^\Lambda \cap U$, $\omega_f(x) = \Lambda$ and $x \notin \Lambda$. So $x \notin \omega_f(x)$, alternatively, $x \in NR(f)$. Consequently,

$$G_K^\Lambda \cap U \subseteq G_K^{\Lambda, N}. \quad (5.50)$$

Therefore, (5.49) and (5.50) produce (5.48). \square

5.3 Non-transitive Case

Proposition B. *Suppose (X, f) is topological expanding and transitive. For any nonempty compact connect subset $K \subseteq M(f, X)$, define $C_K := \overline{\bigcup_{\mu \in K} S_\mu}$. Then if $C_K \neq X$, we have*

$$h_{\text{top}}(G_K^N) = h_{\text{top}}(G_K) = \inf\{h_\mu : \mu \in K\}.$$

Proof. Since $h_{\text{top}}(G_K^N) \leq h_{\text{top}}(G_K) \leq \inf\{h_\mu : \mu \in K\}$ by lemma 2.1, it is left to show that

$$h_{\text{top}}(G_K^N) \geq \inf\{h_\mu : \mu \in K\}. \quad (5.51)$$

Indeed, by lemma 2.11, there exists an $x \in X$ such that

$$\overline{\bigcup_{y \in \omega_f(x)} \omega_f(y)} \subseteq C_K \subseteq \omega_f(x) \neq X.$$

Then by lemma 2.5, one sees that

$$M(f, \omega_f(x)) = M(f, C_K).$$

In particular, $K \subseteq M(f, \omega_f(x))$. Moreover, $\omega_f(x)$ is chain-transitive [37], so by proposition A and proposition 5.12, we see that

$$h_{\text{top}}(G_K^{\omega_f(x)} \cap NR(f)) = \inf\{h_\mu : \mu \in K\}.$$

Finally, it is clear that $G_K^N \supseteq (G_K^{\omega_f(x)} \cap NR(f))$, proving (5.51). \square

6 Multi-fractal Analysis

Let $\Theta_S := \{\Lambda \mid \Lambda \subsetneq X \text{ and } f|_\Lambda \text{ has shadowing}\}$, $\Theta_T := \{\Lambda \mid \Lambda \subsetneq X \text{ and } f|_\Lambda \text{ is transitive}\}$ and $\Theta = \Theta_T \cap \Theta_S$. Throughout this section, we consider a topological dynamical system (X, f) and suppose

1. $\Lambda \subsetneq X$ is a closed f -invariant set;
2. φ is a continuous function on X .

Define $I_\varphi^\Lambda(f) = \{x \in I_\varphi(f) \mid V_f(x) \subseteq M(f, \Lambda)\}$. Denote

$$L_\varphi^\Lambda = \left[\inf_{\mu \in M(f, \Lambda)} \int \varphi d\mu, \sup_{\mu \in M(f, \Lambda)} \int \varphi d\mu \right] = [L_1^\Lambda, L_2^\Lambda].$$

For $a \in L_\varphi^\Lambda$, define $R_\varphi^\Lambda(a) = \{x \in R_\varphi(a) \mid V_f(x) \subseteq M(f, \Lambda)\}$ and denote

$$t_a^\Lambda = \sup_{\mu \in M(f, \Lambda)} \{h_\mu : \int \varphi d\mu = a\}. \quad (6.52)$$

6.1 Useful facts

The following lemma is not hard to check.

Lemma 6.1. If $I_\varphi^\Lambda(f) \neq \emptyset$, then

$$\inf_{\mu \in M(f, \Lambda)} \int \varphi d\mu < \sup_{\mu \in M(f, \Lambda)} \int \varphi d\mu. \quad (6.53)$$

Lemma 6.2. Suppose there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. Then

$$\text{Int}(L_\varphi^\Lambda) \subseteq \left\{ \int \varphi d\mu : \mu \in M(f, \Lambda) \text{ and } S_\mu = C_\Lambda \right\}.$$

Proof. For any $a \in \text{Int}(L_\varphi^\Lambda)$, we need to find a $\mu \in M(f, \Lambda)$ with $S_\mu = C_\Lambda$ such that $\int \varphi d\mu = a$. Indeed, if $\int \varphi d\lambda = a$, then choose $\mu = \lambda$. Otherwise, we suppose without loss of generality that $\int \varphi d\lambda < a < L_2^\Lambda$. Then there exists a $\theta \in (0, 1)$ such that

$$\theta \int \varphi d\lambda + (1 - \theta)L_2^\Lambda = a.$$

Now $\mu = \theta\lambda + (1 - \theta)\mu_{\max}$ suffices for our needs. \square

Definition 6.3. We say (Λ, f) satisfies the center-fat property if for any $k \in \mathbb{N}$ and k f -invariant measures $\{\mu_i\}_{i=1}^k$ on Λ with each $S_{\mu_i} \neq C_\Lambda$, one has $\bigcup_{i=1}^k S_{\mu_i} \neq C_\Lambda$.

Lemma 6.4. If $(\Lambda, f|_\Lambda)$ is topologically transitive, then for any closed f -invariant set $\Lambda_i \subsetneq \Lambda$, $1 \leq i \leq n$, one has $\bigcup_{i=1}^n \Lambda_i \neq \Lambda$. In particular, If $(\Lambda, f|_\Lambda)$ is topologically transitive and there is a $\mu \in M(f, \Lambda)$ such that $S_\mu = C_\Lambda$, then (Λ, f) has the center-fat property.

Proof. Since $(\Lambda, f|_\Lambda)$ is transitive, there is an $x \in \Lambda$ such that $\overline{\text{orb}(x, f)} = \Lambda$. If $\bigcup_{i=1}^n \Lambda_i = \Lambda$, then x locates in some closed Λ_j with $1 \leq j \leq n$. This implies that $\overline{\text{orb}(x, f)}$ coincides with Λ_j , contradicting the fact that $\Lambda_j \neq \Lambda$. \square

Lemma 6.5. Let K be a nonempty compact connected set of $M(f, \Lambda)$.

(1) If $\inf\{\int \varphi d\mu : \mu \in K\} < \sup\{\int \varphi d\mu : \mu \in K\}$, then $G_K \subset I_\varphi^\Lambda(f)$.

(2) If $\inf\{\int \varphi d\mu : \mu \in K\} = \sup\{\int \varphi d\mu : \mu \in K\} = a$, then $G_K \subset R_\varphi^\Lambda(a)$.

Proof. (1) Since $G_K = \{x \in X : V_f(x) = K\}$ and $K \subseteq M(f, \Lambda)$, $G_K \subseteq \{x \in X : V_f(x) \subseteq M(f, \Lambda)\}$. So it is left to show that $G_K \subseteq I_\varphi(f)$. Indeed, choose $\mu_1, \mu_2 \in K$ such that $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. Then for any $x \in G_K$, there exists two sequences $m_k \rightarrow \infty$ and $n_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} \mathcal{E}_{m_k}(x) = \mu_1$ and $\lim_{k \rightarrow \infty} \mathcal{E}_{n_k}(x) = \mu_2$. Consequently,

$$\lim_{k \rightarrow \infty} \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i x) = \int \varphi d\mu_1 \neq \int \varphi d\mu_2 = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i x).$$

This implies that $x \in I_{\varphi|_\Lambda}(f|_\Lambda)$ and thus $G_K \subset I_{\varphi|_\Lambda}(f|_\Lambda)$.

(2) Since $G_K = \{x \in X : V_f(x) = K\}$ and $K \subseteq M(f, \Lambda)$, $G_K \subseteq \{x \in X : V_f(x) \subseteq M(f, \Lambda)\}$. So it is left to show that $G_K \subseteq I_\varphi(f)$. Indeed, for any $x \in G_K$, consider any sequence $l_k \rightarrow \infty$ such that $\frac{1}{l_k} \sum_{i=0}^{l_k-1} \varphi(f^i x)$ converges as $k \rightarrow \infty$. Then there is a subsequence l_{k_p} of l_k such that $\mathcal{E}_{l_{k_p}}$ converges to some $\nu \in K$ as $p \rightarrow \infty$. Consequently,

$$\lim_{k \rightarrow \infty} \frac{1}{l_k} \sum_{i=0}^{l_k-1} \varphi(f^i x) = \lim_{p \rightarrow \infty} \frac{1}{l_{k_p}} \sum_{i=0}^{l_{k_p}-1} \varphi(f^i x) = \int \varphi d\nu = a.$$

Therefore, $x \in R_{\varphi|_\Lambda}(a)$ and thus $G_K \subset R_{\varphi|_\Lambda}(a)$. \square

6.2 Variational Principle

Proposition 6.6. Let $\Lambda \subseteq X$ be closed and f -invariant. Moreover, (Λ, f) satisfies the star-saturated property. If $\varphi \in C(X)$ with $I_\varphi^\Lambda \neq \emptyset$ and $a \in \text{Int}(L_\varphi^\Lambda)$, then

$$(1) \quad h_{\text{top}}(f, I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda).$$

$$(2) \quad h_{\text{top}}(f, R_\varphi^\Lambda(a)) = t_a^\Lambda.$$

Proof. (1) For any $\eta > 0$, we use the variational principle to find an ergodic measure $\mu \in M(f, \Lambda)$ such that $h_\mu > h_{\text{top}}(f|_\Lambda) - \eta$. Since $I_\varphi^\Lambda \neq \emptyset$, there is another $\nu \in M(f, \Lambda)$ with $\int \varphi d\mu \neq \int \varphi d\nu$. Now choose a $t \in (0, 1)$ close to 1 such that $\omega = t\mu + (1-t)\nu$ satisfies that

$$h_\omega \geq th_\mu > h_{\text{top}}(f|_\Lambda).$$

Let $K := \text{cov}\{\mu, \omega\}$. Then lemma 6.5 indicates that $G_K^\Lambda \subseteq I_\varphi^\Lambda(f)$. So the star-saturated property of (Λ, f) implies that

$$h_{\text{top}}(f, I_\varphi^\Lambda(f)) \geq h_{\text{top}}(f, G_K^\Lambda) = \min\{h_\mu, h_\omega\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we see

$$h_{\text{top}}(f, I_\varphi^\Lambda(f)) \geq h_{\text{top}}(f|_\Lambda).$$

Moreover, for any $x \in I_\varphi^\Lambda(f)$, $V_f(x) \subset M(f, \Lambda)$. So by lemma 2.1, we see

$$h_{\text{top}}(f, I_\varphi^\Lambda(f)) \leq \sup\{h_\kappa : \kappa \in V_f(x)\} \leq \sup\{h_\alpha : \alpha \in M(f, \Lambda)\} = h_{\text{top}}(f|_\Lambda).$$

The proof is completed.

(2) For any $\eta > 0$, we choose by definition a $\mu \in M(f, \Lambda)$ with $\int \varphi d\mu = a$ such that $h_\mu > t_a^\Lambda - \eta$. Let $K := \{\mu\}$. Then lemma 6.5 indicates that $G_K^\Lambda \subseteq R_\varphi^\Lambda(a)$. So the star-saturated property of (Λ, f) implies that

$$h_{\text{top}}(f, R_\varphi^\Lambda(a)) \geq h_{\text{top}}(f, G_K^\Lambda) = h_\mu > t_a^\Lambda - \eta.$$

By the arbitrariness of η , we see

$$h_{\text{top}}(f, R_\varphi^\Lambda(a)) \geq t_a^\Lambda.$$

Moreover, for any $x \in R_\varphi^\Lambda(a)$, $V_f(x) \subset M(f, \Lambda)$. So by lemma 2.1, we see

$$h_{\text{top}}(f, R_\varphi^\Lambda(a)) \leq \sup\{h_\kappa : \kappa \in V_f(x), \int \varphi d\kappa = a\} \leq \sup\{h_\alpha : \alpha \in M(f, \Lambda), \int \varphi d\alpha = a\} = t_a^\Lambda.$$

The proof is completed. □

Corollary 6.7. Let (X, f) be topologically expanding (resp. topologically hyperbolic) and transitive. If $\varphi \in C(X)$ with $I_\varphi \neq \emptyset$ and $a \in \text{Int}(L_\varphi)$, then

$$(1) \quad h_{\text{top}}(f, I_\varphi(f)) = h_{\text{top}}(f).$$

$$(2) \quad h_{\text{top}}(f, R_\varphi(a)) = t_a.$$

Proof. This follows from proposition 4.8, proposition A and proposition 6.6. □

6.3 Auxiliary Propositions

Proposition 6.8. Suppose there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $I_\varphi^\Lambda(f) \neq \emptyset$, then for any $\eta > 0$ and any $n \in \mathbb{N}$, there exist f -invariant measures $\{\lambda_k\}_{k=1}^n$ on Λ such that

- (1) $S_{\lambda_k} = C_\Lambda, k = 1, \dots, n$.
- (2) $h_{\lambda_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, \dots, n$.
- (3) $\int \varphi d\lambda_1 < \int \varphi d\lambda_2 < \dots < \int \varphi d\lambda_n$.

Proof. By the variational principle, we choose a $\nu \in M_{\text{erg}}(f, \Lambda)$ such that $h_\nu > h_{\text{top}}(f|_\Lambda) - \eta$. Then we choose $\theta \in (0, 1)$ close to 1 such that $\omega := \theta\nu + (1 - \theta)\lambda$ satisfies that

$$h_\omega = \theta h_\nu + (1 - \theta)h_\lambda \geq \theta h_\nu > h_{\text{top}}(f|_\Lambda) - \eta.$$

Meanwhile, since $I_\varphi^\Lambda(f) \neq \emptyset$, we obtain by lemma 6.1 two f -invariant measure τ and κ on Λ such that $\int \varphi d\tau < \int \varphi d\kappa$. Then we choose an arbitrary sequence $0 < c_1 < c_2 < \dots < c_n < 1$ and let

$$\mu_k = c_k \kappa + (1 - c_k)\tau, \quad k = 1, \dots, n.$$

Further, we choose $\tilde{\theta} \in (0, 1)$ close to 1 such that $\lambda_k = \tilde{\theta}\omega + (1 - \tilde{\theta})\mu_k$ satisfy that

$$h_{\lambda_k} = \tilde{\theta}h_\omega + (1 - \tilde{\theta})h_{\mu_k} \geq \tilde{\theta}h_\omega > h_{\text{top}}(f|_\Lambda) - \eta, \quad k = 1, \dots, n.$$

It is left to show that the $\{\lambda_k\}_{k=1}^n$ constructed above satisfy property (1) and (3). Indeed, $S_\omega = S_\nu \cup S_\lambda = C_\Lambda$ and $S_{\lambda_k} = S_\omega \cup S_{\mu_k} = C_\Lambda, k = 1, \dots, n$. Moreover, for $k = 1, \dots, n - 1$,

$$\int \varphi d\lambda_{k+1} - \int \varphi d\lambda_k = (1 - \tilde{\theta}) \left(\int \varphi d\mu_{k+1} - \int \varphi d\mu_k \right) = (1 - \tilde{\theta})(c_{k+1} - c_k) \left(\int \varphi d\kappa - \int \varphi d\tau \right) > 0.$$

□

Proposition 6.9. Suppose $M_{\text{erg}}(f, \Lambda)$ is star-entropy-dense in $M(f, \Lambda)$ and there exists a $\lambda \in M(f, \Lambda)$ with $S_\lambda = C_\Lambda$. If $I_\varphi^\Lambda(f) \neq \emptyset$, then for any $\eta > 0$ and any $n \in \mathbb{N}$, there exist n f -invariant measures $\{\lambda'_k\}_{k=1}^n$ on Λ such that

- (a) $\{S_{\lambda'_k}\}_{k=1}^n$ are pairwise disjoint and consequently, $S_{\lambda'_k} \neq C_\Lambda, k = 1, \dots, n$.
- (b) $h_{\lambda'_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, \dots, n$.
- (c) $\int \varphi d\lambda'_1 < \int \varphi d\lambda'_2 < \dots < \int \varphi d\lambda'_n$.

Proof. For any $\eta > 0$ and any $n \in \mathbb{N}$, by proposition 6.8, we choose $\{\lambda_k\}_{k=1}^n \subseteq M(f, \Lambda)$ satisfying the properties (1)(2)(3) there. Let

$$\varepsilon := \min \left\{ \int \varphi d\lambda_{k+1} - \int \varphi d\lambda_k : k = 1, \dots, n - 1 \right\} > 0.$$

By the star-entropy-dense property, if we let $G_{\lambda_k} := \{\omega \in M(f, \Lambda) : |\int \varphi d\omega - \int \varphi d\lambda_k| < \varepsilon/3\}$ for each $k = 1, \dots, n$, then there exists $\lambda'_k \in M_{\text{erg}}(f, \Lambda)$ with

- (P1) $M(f, S_{\lambda'_k}) \subseteq G_{\lambda_k}, k = 1, \dots, n$.
- (P2) $h_{\lambda'_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, \dots, n$.

Note that $\{G_{\lambda_k}\}_{k=1}^n$ are pairwise disjoint. So (P1) indicates that $\{M(f, S_{\lambda'_k})\}_{k=1}^n$ are pairwise disjoint. As a result, $\{S_{\lambda'_k}\}_{k=1}^n$ are pairwise disjoint and consequently, $S_{\lambda'_k} \neq C_\Lambda, k = 1, \dots, n$.

Finally, property (3) of $\{\lambda_k\}_{k=1}^n$ and the selections of $\{G_k\}_{k=1}^n$ yield property (c) of $\{\lambda'_k\}_{k=1}^n$. \square

Proposition 6.10. Suppose $M_{erg}(f, \Lambda)$ is star-entropy-dense in $M(f, \Lambda)$ and there exists a $\lambda \in M(f, \Lambda)$ with $S_\lambda = C_\Lambda$. If $(\Lambda, f|_\Lambda)$ satisfies the center-fat property and $I_\varphi^\Lambda(f) \neq \emptyset$, then for any $\eta > 0$ and any $n \in \mathbb{N}$, there exist $\{\bar{\lambda}_k\}_{k=1}^n \subseteq M(f, \Lambda)$ such that

- (i) $S_{\bar{\lambda}_1} = S_{\bar{\lambda}_2} = \dots = S_{\bar{\lambda}_n} \neq C_\Lambda$.
- (ii) $h_{\bar{\lambda}_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, \dots, n$.
- (iii) $\int \varphi d\bar{\lambda}_1 < \int \varphi d\bar{\lambda}_2 < \dots < \int \varphi d\bar{\lambda}_n$.

Proof. Choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ which satisfy properties (1)(2)(3) in proposition 6.9. Then select an arbitrary sequence $0 < t_1 < t_2 < \dots < t_n < 1$ and let

$$\bar{\lambda}_k := t_k \lambda'_2 + (1 - t_k) \lambda'_1, k = 1, \dots, n.$$

Since (Λ, f) satisfies the center-fat property and $S_{\lambda'_k} \neq C_\Lambda, k = 1, \dots, n$,

$$S_{\bar{\lambda}_1} = \dots = S_{\bar{\lambda}_n} = S_{\lambda'_1} \cup S_{\lambda'_2} \neq C_\Lambda.$$

Moreover,

$$h_{\bar{\lambda}_k} \geq \min\{h_{\lambda'_1}, h_{\lambda'_2}\} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, \dots, n.$$

Finally,

$$\int \varphi d\bar{\lambda}_{k+1} - \int \varphi d\bar{\lambda}_k = (t_{k+1} - t_k) \left(\int \varphi d\lambda'_2 - \int \varphi d\lambda'_1 \right) > 0, k = 1, \dots, n-1.$$

\square

Proposition 6.11. Suppose there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_a^\Lambda)$, then for any $\eta > 0$, there exists a $\tau \in M(f, \Lambda)$ such that

- (1) $S_\tau = C_\Lambda$.
- (2) $h_\tau > t_a^\Lambda - \eta$.
- (3) $\int \varphi d\tau = a$.

Proof. By the definition of t_a^Λ , there exists a $\nu \in M_{erg}(f, \Lambda)$ with $\int \varphi d\nu = a$ such that $h_\nu > t_a^\Lambda - \eta$. Moreover, by lemma 6.2, there exists a $\mu \in M(f, \Lambda)$ with $S_\mu = C_\Lambda$ such that $\int \varphi d\mu = a$. Now choose $\theta \in (0, 1)$ close to 1 such that $\tau := \theta\nu + (1 - \theta)\mu$ satisfies that

$$h_\tau \geq \theta h_\nu > t_a^\Lambda - \eta.$$

Then $S_\tau = S_\nu \cup S_\mu = C_\Lambda$ and $\int \varphi d\tau = \theta \int \varphi d\nu + (1 - \theta) \int \varphi d\mu = a$. \square

Proposition 6.12. Suppose $M_{erg}(f, \Lambda)$ is star-entropy-dense in $M(f, \Lambda)$ and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_a^\Lambda)$, then for any $\eta > 0$ and $n \in \mathbb{N}$, there exists $\lambda'_1, \dots, \lambda'_n \in M(f, \Lambda)$ such that

- (a) $\{S_{\lambda'_k}\}_{k=1}^n$ are pairwise disjoint and consequently, $S_{\lambda'_k} \neq C_\Lambda, k = 1, \dots, n$.

(b) $h_{\lambda'_k} > t_a^\Lambda - \eta, k = 1, \dots, n.$

(c) $\int \varphi d\lambda'_k = a, k = 1, \dots, n.$

Proof. First, choose a $\kappa \in M(f, \Lambda)$ with $\int \varphi d\kappa = a$ such that $h_\kappa > t_a^\Lambda - \eta$. Then choose $0 < \theta_1 < \dots < \theta_n < 1$ close to 1 such that $\mu_k := \theta_k \nu + (1 - \theta_k) \mu_{\min}$ and $\nu_k := \theta_k \nu + (1 - \theta_k) \mu_{\max}, k = 1, \dots, n$ satisfy that

$$h_{\mu_k} \geq \theta h_\kappa > t_a^\Lambda \text{ and } h_{\nu_k} \geq \theta h_\kappa > t_a^\Lambda.$$

Let

$$\varepsilon := \min \left\{ \int \varphi d\mu_{k+1} - \int \varphi d\mu_k, \int \varphi d\nu_k - \int \varphi d\nu_{k+1}, a - \int \varphi d\mu_n, \int \varphi d\nu_n - a : k = 1, \dots, n-1 \right\}.$$

Consider the following neighborhoods of μ_k and ν_k :

$$F_{\mu_k} := \left\{ \xi \in M(f, \Lambda) : \left| \int \varphi d\xi - \int \varphi d\mu_k \right| < \varepsilon/3 \right\} \text{ and } F_{\nu_k} := \left\{ \xi \in M(f, \Lambda) : \left| \int \varphi d\xi - \int \varphi d\nu_k \right| < \varepsilon/3 \right\}.$$

Then by the star-entropy-dense property, one obtain $2n$ ergodic measures $\mu'_k, \nu'_k \in M(f, \Lambda), k = 1, \dots, n$ such that

1. $M(f, S_{\mu'_k}) \subseteq F_{\mu_k}$ and $M(f, S_{\nu'_k}) \subseteq F_{\nu_k}, k = 1, \dots, n.$
2. $h_{\mu'_k} > t_a^\Lambda - \eta$ and $h_{\nu'_k} > t_a^\Lambda - \eta, k = 1, \dots, n.$

Note by the selection of ε and F_{μ_k}, F_{ν_k} that $M(f, S_{\mu'_k}), M(f, S_{\nu'_k}), k = 1, \dots, n$ are pairwise disjoint. So $S_{\mu'_k}, S_{\nu'_k}, k = 1, \dots, n$ are pairwise disjoint. Note also that $\int \varphi d\mu'_k < a < \int \varphi d\nu'_k$ for each $k = 1, \dots, n$. Thus we can choose $c_k \in (0, 1)$ such that $c_k \int \varphi d\mu'_k + (1 - c_k) \int \varphi d\nu'_k = a$ for each $k = 1, \dots, n$. Now let $\lambda'_k = c_k \mu'_k + (1 - c_k) \nu'_k$, then

- (i) $S_{\lambda'_k} = S_{\mu'_k} \cup S_{\nu'_k}, k = 1, \dots, n$ are pairwise disjoint and consequently, $S_{\lambda_k} \neq C_\Lambda, k = 1, \dots, n.$
- (ii) $h_{\lambda'_k} \geq \min\{h_{\mu'_k}, h_{\nu'_k}\} > t_a^\Lambda - \eta, k = 1, \dots, n.$
- (iii) $\int \varphi d\lambda'_k = c_k \int \varphi d\mu'_k + (1 - c_k) \int \varphi d\nu'_k = a, k = 1, \dots, n.$

□

Proposition 6.13. Suppose $M_{erg}(f, \Lambda)$ is star-entropy-dense in $M(f, \Lambda)$ and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $(\Lambda, f|_\Lambda)$ has the center-fat property and $a \in \text{Int}(L_a^\Lambda)$, then for any $\eta > 0$ and $n \in \mathbb{N}$, there exist n distinct measures $\bar{\lambda}_1, \dots, \bar{\lambda}_n \in M(f, \Lambda)$ such that

- (a) $S_{\bar{\lambda}_1} = S_{\bar{\lambda}_2} = \dots = S_{\bar{\lambda}_n} \neq C_\Lambda.$
- (b) $h_{\bar{\lambda}_k} > t_a^\Lambda - \eta, k = 1, \dots, n.$
- (c) $\int \varphi d\bar{\lambda}_k = a, k = 1, \dots, n.$

Proof. Choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ which satisfy properties (a)(b)(c) in proposition 6.12. Then select an arbitrary sequence $0 < t_1 < t_2 < \dots < t_n < 1$ and let

$$\bar{\lambda}_k := t_k \lambda'_2 + (1 - t_k) \lambda'_1, k = 1, \dots, n.$$

Since (Λ, f) satisfies the center-fat property and $S_{\lambda'_k} \neq C_\Lambda, k = 1, \dots, n,$

$$S_{\bar{\lambda}_1} = \dots = S_{\bar{\lambda}_n} = S_{\lambda'_1} \cup S_{\lambda'_2} \neq C_\Lambda.$$

Moreover,

$$h_{\bar{\lambda}_k} \geq \min\{h_{\lambda'_1}, h_{\lambda'_2}\} > h_{\text{top}}(f|_{\Lambda}) - \eta, k = 1, \dots, n.$$

Finally,

$$\int \varphi d\bar{\lambda}_k = t_k \int \varphi d\lambda'_1 + (1 - t_k) \int \varphi d\lambda'_2 = a, \quad k = 1, \dots, n.$$

□

6.4 Multi-fractal Analysis for the Irregular Set

For $\Lambda \subseteq X$, define $\Xi_{\Lambda} := \{S_{\mu} \mid S_{\mu} \subsetneq \Lambda, \mu \in M(f, X)\}$.

Theorem 6.14. Suppose $(\Lambda, f|_{\Lambda})$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_{\lambda} = C_{\Lambda}$. If $I_{\varphi}^{\Lambda}(f) \neq \emptyset$, then

- (1) $h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda})$;
- (2) under the additional condition that $(\Lambda, f|_{\Lambda})$ satisfies the center-fat property and the star-entropy-dense property, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda});$$

- (3) under the additional condition that $(\Lambda, f|_{\Lambda})$ satisfies the star-entropy-dense property, one has for any $Z \in \Xi_{\Lambda} \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda});$$

- (4) under the additional condition that $(\Lambda, f|_{\Lambda})$ satisfies the center-fat property and the star-entropy-dense property, one has for any $Z \in \Xi_{\Lambda} \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda});$$

Proof. In fact, it is sufficient to prove the subsequent Propositions 6.15-6.18.

Proposition 6.15. Suppose $(\Lambda, f|_{\Lambda})$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_{\lambda} = C_{\Lambda}$. If $I_{\varphi}^{\Lambda}(f) \neq \emptyset$, then

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda}).$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

For any $\eta > 0$, choose $\lambda_1, \lambda_2 \in M(f, \Lambda)$ satisfying (1)(2)(3) in proposition 6.8. Define $K := \text{cov}\{\lambda_1, \lambda_2\}$. Observe that $\bigcap_{\mu \in K} S_{\mu} = C_{\Lambda}$ which yields by lemma 5.2 that

$$G_K^{\Lambda} \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, $\int \varphi d\lambda_1 \neq \int \varphi d\lambda_2$ indicates by lemma 6.5 that $G_K^{\Lambda} \subseteq G_K \subseteq I_{\varphi}^{\Lambda}(f)$. So we have

$$G_K^{\Lambda, N} \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f). \quad (6.54)$$

Finally, since $(\Lambda, f|_{\Lambda})$ is nonrecurrently-star-saturated,

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_{\kappa} : \kappa \in K\} = \min\{h_{\lambda_1}, h_{\lambda_2}\} > h_{\text{top}}(f|_{\Lambda}) - \eta.$$

By the arbitrariness of η , we obtain our result. □

Proposition 6.16. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. Moreover, suppose additionally that $(\Lambda, f|_\Lambda)$ satisfies the center-fat property and the star-entropy-dense property. If $I_\varphi^\Lambda(f) \neq \emptyset$, then

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda);$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

For any $\eta > 0$, choose $\overline{\lambda}_1, \overline{\lambda}_2 \in M(f, \Lambda)$ satisfying (i)(ii)(iii) in proposition 6.10. Define $K := \text{cov}\{\overline{\lambda}_1, \overline{\lambda}_2\}$. Let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.55)$$

Indeed, by property (i) of $\overline{\lambda}_1, \overline{\lambda}_2$, $\bigcap_{\tau \in K} S_\tau = \overline{\bigcup_{\tau \in K} S_\tau} \subsetneq C_\Lambda$. So by lemma 5.2, one obtains that

$$G_K^\Lambda \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

By property (iii) of $\overline{\lambda}_1, \overline{\lambda}_2$ and lemma 6.5, we see $G_K^\Lambda \subseteq G_K \subset I_\varphi^\Lambda(f)$. Therefore, (6.55) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\overline{\lambda}_1}, h_{\overline{\lambda}_2}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result. \square

Proposition 6.17. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. Moreover, suppose additionally that $(\Lambda, f|_\Lambda)$ satisfies the star-entropy-dense property. If $I_\varphi^\Lambda(f) \neq \emptyset$, then one has for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda);$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

Case 1: $Z = \emptyset$.

For any $\eta > 0$, choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ satisfying (a)(b)(c) in proposition 6.9. Moreover, choose λ_3 satisfying (1)(2) in proposition 6.8 and set $\lambda'_3 = \lambda_3$. Define $K := \text{cov}\{\lambda'_1, \lambda'_2, \lambda'_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.56)$$

Indeed, by proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\lambda'_1} \cap S_{\lambda'_2} \cap S_{\lambda'_3} = \emptyset.$$

Meanwhile, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\lambda'_1} \cup S_{\lambda'_2} \cup S_{\lambda'_3} = C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} = C_\Lambda$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, since $\int \varphi d\lambda'_1 \neq \int \varphi d\lambda'_2$ and lemma 6.5, we see $G_K^\Lambda \subseteq G_K \subset I_\varphi^\Lambda(f)$. Therefore, (6.56) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\lambda'_1}, h_{\lambda'_2}, h_{\lambda'_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result.

Case 2: $Z \in \Xi_\Lambda$.

Suppose $Z = S_\mu \subsetneq \Lambda$ for some $\mu \in M(f, X)$. For any $\eta > 0$, choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ satisfying (a)(b)(c) in proposition 6.9. Then choose $\theta \in (0, 1)$ close to 1 such that $\widehat{\lambda}_k = \theta\lambda'_k + (1 - \theta)\mu, k = 1, 2$ satisfy that

$$h_{\widehat{\lambda}_k} = \theta h_{\lambda'_k} + (1 - \theta)h_\mu \geq \theta h_{\lambda'_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, 2.$$

Moreover, choose λ_3 satisfying (1)(2) in proposition 6.8 and set $\widehat{\lambda}_3 = \lambda_3$. Define $K := \text{cov}\{\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.57)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\widehat{\lambda}_1} \cap S_{\widehat{\lambda}_2} \cap S_{\widehat{\lambda}_3} = S_{\widehat{\lambda}_1} \cap S_{\widehat{\lambda}_2} = S_\mu.$$

Meanwhile, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\widehat{\lambda}_1} \cup S_{\widehat{\lambda}_2} \cup S_{\widehat{\lambda}_3} = C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} = C_\Lambda$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, since $\int \varphi d\lambda'_1 \neq \int \varphi d\lambda'_2$, $\int \varphi d\widehat{\lambda}_1 \neq \int \varphi d\widehat{\lambda}_2$. So by lemma 6.5, we see $G_K^\Lambda \subseteq G_K \subset I_\varphi^\Lambda(f)$. Therefore, (6.59) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\widehat{\lambda}_1}, h_{\widehat{\lambda}_2}, h_{\widehat{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result. \square

Proposition 6.18. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. Moreover, suppose additionally that $(\Lambda, f|_\Lambda)$ satisfies the center-fat property and the star-entropy-dense property. If $I_\varphi^\Lambda(f) \neq \emptyset$, then one has for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda);$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

Case 1: $Z = \emptyset$.

For any $\eta > 0$, choose $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3 \in M(f, \Lambda)$ satisfying (i)(ii)(iii) in proposition 6.10. Define $K := \text{cov}\{\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.58)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\overline{\lambda}_1} \cap S_{\overline{\lambda}_2} \cap S_{\overline{\lambda}_3} = \emptyset.$$

Meanwhile, since (Λ, f) has the center-fat property, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\overline{\lambda}_1} \cup S_{\overline{\lambda}_2} \cup S_{\overline{\lambda}_3} \subsetneq C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} \subsetneq C_\Lambda$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, since $\int \varphi d\overline{\lambda}_1 \neq \int \varphi d\overline{\lambda}_2$ and lemma 6.5, we see $G_K^\Lambda \subseteq G_K \subset I_\varphi^\Lambda(f)$. Therefore, (6.56) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\overline{\lambda}_1}, h_{\overline{\lambda}_2}, h_{\overline{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result.

Case 2: $Z \in \Xi_\Lambda$.

Suppose $Z = S_\mu \subsetneq \Lambda$ for some $\mu \in M(f, X)$. For any $\eta > 0$, choose $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \in M(f, \Lambda)$ satisfying (i)(i)(iii) in proposition 6.10. Now choose $\theta \in (0, 1)$ close to 1 such that $\hat{\lambda}_k = \theta \bar{\lambda}_k + (1 - \theta)\mu, k = 1, 2, 3$ satisfy that

$$h_{\hat{\lambda}_k} = \theta h_{\bar{\lambda}_k} + (1 - \theta)h_\mu \geq \theta h_{\bar{\lambda}_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, 2, 3.$$

Define $K := \text{cov}\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.59)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\kappa \in V_f(x)} S_\kappa = \bigcap_{\kappa \in K} S_\kappa = S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} \cap S_{\hat{\lambda}_3} = (S_{\bar{\lambda}_1} \cap S_{\bar{\lambda}_2} \cap S_{\bar{\lambda}_3}) \cup S_\mu = S_\mu.$$

Meanwhile, since (Λ, f) satisfies the center-fat property and $S_{\bar{\lambda}_k} \neq \Lambda, k = 1, 2, 3$, one has $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\bar{\lambda}_1} \cup S_{\bar{\lambda}_2} \cup S_{\bar{\lambda}_3} \subsetneq C_\Lambda$. Moreover, $S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} = S_{\bar{\lambda}_1} \cap S_{\bar{\lambda}_2} = \emptyset$. So $\bigcap_{\kappa \in K} S_\kappa \subsetneq \overline{\bigcup_{\kappa \in K} S_\kappa}$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, since $|\int \varphi d\hat{\lambda}_1 - \int \varphi d\hat{\lambda}_2| = \theta |\int \varphi d\bar{\lambda}_1 - \int \varphi d\bar{\lambda}_2| \neq 0$, we see by lemma 6.5 that $G_K^\Lambda \subseteq G_K \subset I_\varphi^\Lambda(f)$. Therefore, (6.59) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\hat{\lambda}_1}, h_{\hat{\lambda}_2}, h_{\hat{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result. □

□

6.5 Multifractal analysis for the Level Set

For $\Lambda \subseteq X$, define $\Xi_\Lambda := \{S_\mu \mid S_\mu \subsetneq \Lambda, \mu \in M(f, X)\}$.

Theorem 6.19. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_\varphi^\Lambda)$, then

(1) $h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$

(2) under the additional condition that $\Lambda, f|_\Lambda$ satisfies the center-fat property, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

(3) for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

(4) under the additional condition that $(\Lambda, f|_\Lambda)$ satisfies the center-fat property, one has for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

Porof. Proposition 6.20-6.23 below lead to the proof of theorem 6.19.

Proposition 6.20. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_\varphi^\Lambda)$, then

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda.$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

For any $\eta > 0$, choose τ satisfying (1)(2)(3) in proposition 6.11. Define $K := \{\tau\}$. Observe that $\bigcap_{\mu \in K} S_\mu = C_\Lambda$ which yields by lemma 5.2 that

$$G_K^\Lambda \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, by lemma 6.5, $G_K^\Lambda \subseteq G_K \subseteq R_\varphi^\Lambda(a)$. So we have

$$G_K^{\Lambda, N} \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(f). \quad (6.60)$$

Finally, since $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated,

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\kappa : \kappa \in K\} = \min\{h_{\lambda_1}, h_{\lambda_2}\} > t_a^\Lambda - \eta.$$

By the arbitrariness of η , we obtain our result. \square

Proposition 6.21. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_\varphi^\Lambda)$, then under the additional condition that $\Lambda, f|_\Lambda$ satisfies the center-fat property, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

For any $\eta > 0$, choose $\overline{\lambda}_1, \overline{\lambda}_2 \in M(f, \Lambda)$ satisfying (i)(ii)(iii) in proposition 6.13. Define $K := \text{cov}\{\overline{\lambda}_1, \overline{\lambda}_2\}$. Let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.61)$$

Indeed, by property (i) of $\overline{\lambda}_1, \overline{\lambda}_2$, $\bigcap_{\tau \in K} S_\tau = \overline{\bigcup_{\tau \in K} S_\tau} \subsetneq C_\Lambda$. So by lemma 5.2, one obtains that

$$G_K^\Lambda \subseteq \{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

By property (iii) of $\overline{\lambda}_1, \overline{\lambda}_2$, we see

$$\int \varphi d(t\overline{\lambda}_1 + (1-t)\overline{\lambda}_2) = t \int \varphi d\overline{\lambda}_1 + (1-t) \int \varphi d\overline{\lambda}_2 = a \text{ for any } 0 \leq t \leq 1.$$

So lemma 6.5, we have $G_K^\Lambda \subseteq G_K \subset R_\varphi^\Lambda(a)$. Therefore, (6.61) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\overline{\lambda}_1}, h_{\overline{\lambda}_2}\} > t_a^\Lambda - \eta.$$

By the arbitrariness of η , we obtain our result. \square

Proposition 6.22. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_\varphi^\Lambda)$, then for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

Case 1: $Z = \emptyset$.

For any $\eta > 0$, choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ satisfying (a)(b)(c) in proposition 6.12. Moreover, choose τ satisfying (1)(2) in proposition 6.11 and set $\lambda'_3 = \tau$. Define $K := \text{cov}\{\lambda'_1, \lambda'_2, \lambda'_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.62)$$

Indeed, by proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\lambda'_1} \cap S_{\lambda'_2} \cap S_{\lambda'_3} = \emptyset.$$

Meanwhile, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\lambda'_1} \cup S_{\lambda'_2} \cup S_{\lambda'_3} = C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} = C_\Lambda$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, it is not hard to check that for any $\mu \in K$, $\int \varphi d\mu = a$. So lemma 6.5 yields that $G_K^\Lambda \subseteq G_K \subset R_\varphi^\Lambda(a)$. Therefore, (6.62) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\lambda'_1}, h_{\lambda'_2}, h_{\lambda'_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result.

Case 2: $Z \in \Xi_\Lambda$.

Suppose $Z = S_\mu \subsetneq \Lambda$ for some $\mu \in M(f, X)$. For any $\eta > 0$, choose $\lambda'_1, \lambda'_2 \in M(f, \Lambda)$ satisfying (a)(b)(c) in proposition 6.12. Then choose $\theta \in (0, 1)$ close to 1 such that $\hat{\lambda}_k = \theta\lambda'_k + (1 - \theta)\mu$, $k = 1, 2$ satisfy that

$$h_{\hat{\lambda}_k} = \theta h_{\lambda'_k} + (1 - \theta)h_\mu \geq \theta h_{\lambda'_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, 2.$$

Moreover, choose τ satisfying (1)(2) in proposition 6.11 and set $\hat{\lambda}_3 = \tau$. Define $K := \text{cov}\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.63)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} \cap S_{\hat{\lambda}_3} = S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} = S_\mu.$$

Meanwhile, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\hat{\lambda}_1} \cup S_{\hat{\lambda}_2} \cup S_{\hat{\lambda}_3} = C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} = C_\Lambda$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, it is not hard to check that for any $\mu \in K$, $\int \varphi d\mu = a$. So lemma 6.5 yields that $G_K^\Lambda \subseteq G_K \subset R_\varphi^\Lambda(a)$. Therefore, (6.63) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\hat{\lambda}_1}, h_{\hat{\lambda}_2}, h_{\hat{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result. \square

Proposition 6.23. Suppose $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated and there is a $\lambda \in M(f, \Lambda)$ such that $S_\lambda = C_\Lambda$. If $a \in \text{Int}(L_\varphi^\Lambda)$, then under the additional condition that $(\Lambda, f|_\Lambda)$ satisfies the center-fat property, one has for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$,

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda;$$

Proof. The " \leq " part is ensured by proposition 6.6. Now let us prove the " \geq " part.

Case 1: $Z = \emptyset$.

For any $\eta > 0$, choose $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \in M(f, \Lambda)$ satisfying (i)(ii)(iii) in proposition 6.13. Define $K := \text{cov}\{\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.64)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu = S_{\bar{\lambda}_1} \cap S_{\bar{\lambda}_2} \cap S_{\bar{\lambda}_3} = \emptyset.$$

Meanwhile, since (Λ, f) has the center-fat property, $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\bar{\lambda}_1} \cup S_{\bar{\lambda}_2} \cup S_{\bar{\lambda}_3} \subsetneq C_\Lambda$. So $\bigcap_{\mu \in K} S_\mu \subsetneq \overline{\bigcup_{\nu \in K} S_\nu} \subsetneq C_\Lambda$. According to lemma 5.2, we have

$$G_K^{\Lambda, N} \subset \{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, it is not hard to check that for any $\mu \in K$, $\int \varphi d\mu = a$. So lemma 6.5 yields that $G_K^\Lambda \subseteq G_K \subset R_\varphi^\Lambda(a)$. Therefore, (6.64) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\bar{\lambda}_1}, h_{\bar{\lambda}_2}, h_{\bar{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result.

Case 2: $Z \in \Xi_\Lambda$.

Suppose $Z = S_\mu \subsetneq \Lambda$ for some $\mu \in M(f, X)$. For any $\eta > 0$, choose $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \in M(f, \Lambda)$ satisfying (i)(i)(iii) in proposition 6.13. Now choose $\theta \in (0, 1)$ close to 1 such that $\hat{\lambda}_k = \theta \bar{\lambda}_k + (1 - \theta)\mu, k = 1, 2, 3$ satisfy that

$$h_{\hat{\lambda}_k} = \theta h_{\bar{\lambda}_k} + (1 - \theta)h_\mu \geq \theta h_{\bar{\lambda}_k} > h_{\text{top}}(f|_\Lambda) - \eta, k = 1, 2, 3.$$

Define $K := \text{cov}\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\}$. Now let us prove that

$$G_K^{\Lambda, N} \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f). \quad (6.65)$$

By proposition 3.3, one has for any $x \in G_K$,

$$X_{\underline{d}}(x) = \bigcap_{\kappa \in V_f(x)} S_\kappa = \bigcap_{\kappa \in K} S_\kappa = S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} \cap S_{\hat{\lambda}_3} = (S_{\bar{\lambda}_1} \cap S_{\bar{\lambda}_2} \cap S_{\bar{\lambda}_3}) \cup S_\mu = S_\mu.$$

Meanwhile, since (Λ, f) satisfies the center-fat property and $S_{\bar{\lambda}_k} \neq \Lambda, k = 1, 2, 3$, one has $\overline{\bigcup_{\kappa \in K} S_\kappa} = S_{\bar{\lambda}_1} \cup S_{\bar{\lambda}_2} \cup S_{\bar{\lambda}_3} \subsetneq C_\Lambda$. Moreover, $S_{\hat{\lambda}_1} \cap S_{\hat{\lambda}_2} = S_{\bar{\lambda}_1} \cap S_{\bar{\lambda}_2} = \emptyset$. So $\bigcap_{\kappa \in K} S_\kappa \subsetneq \overline{\bigcup_{\kappa \in K} S_\kappa}$. According to lemma 5.2, we have

$$G_K^\Lambda \subset \{x \in X \mid S_\mu = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$

Moreover, it is not hard to check that for any $\mu \in K$, $\int \varphi d\mu = a$. So lemma 6.5 yields that $G_K^\Lambda \subseteq G_K \subset R_\varphi^\Lambda(a)$. Therefore, (6.65) is proved.

Finally, $(\Lambda, f|_\Lambda)$ is nonrecurrently-star-saturated, so

$$h_{\text{top}}(f, G_K^{\Lambda, N}) = \inf\{h_\mu : \mu \in K\} = \min\{h_{\widehat{\lambda}_1}, h_{\widehat{\lambda}_2}, h_{\widehat{\lambda}_3}\} > h_{\text{top}}(f|_\Lambda) - \eta.$$

By the arbitrariness of η , we obtain our result. □

□

7 Proof of Main Theorems

7.1 Preliminary Lemmas

Lemma 7.1. Suppose (X, f) is topologically expanding and transitive. Let $\varphi : X \rightarrow \mathbb{R}$ be a continuous function and assume that $\text{Int}(L_\varphi) \neq \emptyset$. Then for any $\eta > 0$, there exist three f -invariant subsets Λ and Θ such that

(1) Λ is topologically expanding and transitive with

$$C_\Lambda = \Lambda, \text{Int}(L_\varphi^\Lambda) \neq \emptyset, h_{\text{top}}(f|_\Lambda) > h_{\text{top}}(f) - \eta. \quad (7.66)$$

(2) Θ is internally chain transitive but not topologically transitive and

$$C_\Theta \subsetneq \Theta, \text{Int}(L_\varphi^\Theta) \neq \emptyset, h_{\text{top}}(f|_\Theta) > h_{\text{top}}(f) - \eta. \quad (7.67)$$

Proof. (1) By the variational principle, we obtain a $\mu \in M_{\text{erg}}(f, X)$ with $h_\mu > h_{\text{top}}(f) - \eta/2$. Since $\text{Int}(L_\varphi) \neq \emptyset$, we obtain a $\nu \in M_{\text{erg}}(f, X)$ with $\int \varphi d\nu \neq \int \varphi d\mu$. Then we choose $\theta \in (0, 1)$ close to 1 such that $\omega := \theta\mu + (1 - \theta)\nu$ satisfies that

$$h_\omega = \theta h_\mu + (1 - \theta)h_\nu \geq \theta h_\mu > h_{\text{top}}(f) - \eta/2.$$

Moreover, one note that $|\int \varphi d\omega - \int \varphi d\mu| = (1 - \theta)|\int \varphi d\mu - \int \varphi d\nu| \neq 0$. Now let

$$K := \{t\mu + (1 - t)\omega : 0 \leq t \leq 1\}.$$

By lemma B, we obtain a topologically expanding basic set $\Lambda \subsetneq X$ with two ergodic measures $\tilde{\mu}, \tilde{\omega} \in M(f, \Lambda)$ such that

$$h_{\tilde{\mu}} > h_\mu - \eta/2 > h_{\text{top}}(f) - \eta, \quad h_{\tilde{\omega}} > h_\omega - \eta/2 > h_{\text{top}}(f) - \eta$$

and

$$\left| \int \varphi d\tilde{\mu} - \int \varphi d\mu \right| < \varepsilon/3, \quad \left| \int \varphi d\tilde{\omega} - \int \varphi d\omega \right| < \varepsilon/3,$$

where $\varepsilon := |\int \varphi d\mu - \int \varphi d\omega| > 0$. By the triangle inequality, one sees that $|\int \varphi d\tilde{\mu} - \int \varphi d\tilde{\omega}| > \varepsilon - \varepsilon/3 - \varepsilon/3 = \varepsilon/3$. This implies that $\text{Int}(L_\varphi^\Lambda) \neq \emptyset$. Moreover,

$$h_{\text{top}}(f|_\Lambda) \geq h_{\tilde{\mu}} > h_{\text{top}}(f) - \eta.$$

Besides, since periodic points are dense in Λ , there is a $\nu \in M(f, \Lambda)$ with full support [25]. This implies that $C_\Lambda = \Lambda$.

(2) Choose Λ as defined in (1). Since (X, f) is topologically expanding and transitive, we can find a $z \notin \Lambda$ such that $\omega_f(z) = \Lambda$. Let $A = \text{orb}(z, f) \cup \Lambda$. Note that A is closed and f -invariant. So by lemma 2.11, there is a point $x \in X$ such that

$$A \subseteq \omega_f(x) \subseteq \bigcup_{l=0}^{\infty} f^{-l}A.$$

In particular,

$$\overline{\bigcup_{y \in \omega_f(x)} \omega_f(x)} \subseteq A \subseteq \omega_f(x) \neq X.$$

Let $\Theta = \omega_f(x)$. Then Θ is chain transitive [37]. By lemma 2.5,

$$M(f, \Theta) = M(f, A) = M(f, \Lambda).$$

This implies that $C_{\Theta} = \Lambda \subsetneq \Theta$ and $\text{Int}(L_{\varphi}^{\Theta}) = \text{Int}(L_{\varphi}^{\Lambda}) \neq \emptyset$. It is left to show that Θ is not transitive. Indeed, suppose the opposite is true, then there is an $a \in \omega_f(x)$ such that $\omega_f(a) = \omega_f(x)$. However, $a = f^n b$ for some $n \in \mathbb{N}$ and $b \in A$. In particular, $\omega_f(a) \subseteq \Lambda \subsetneq \omega_f(x)$, a contradiction. \square

Lemma 7.2. Suppose $f : X \rightarrow X$ is transitive and topologically expanding. Let $\varphi : X \rightarrow \mathbb{R}$ be a continuous function and assume that $\text{Int}(L_{\varphi}) \neq \emptyset$. Then for any $a \in \text{Int}(L_{\varphi})$ and any $\eta > 0$, there are three topologically expanding basic set $\Lambda_i \subseteq \Lambda \subsetneq X$ ($i = 1, 2$) such that

(1) $h_{\text{top}}(\Lambda_i) > t_a - \eta$, $i = 1, 2$;

(2)

$$\sup_{\mu \in M_f(\Lambda_1)} \int \varphi(x) d\mu < a < \inf_{\mu \in M_f(\Lambda_2)} \int \varphi(x) d\mu. \quad (7.68)$$

In particular,

$$C_{\Lambda} = \Lambda \text{ and } t_a^{\Lambda} > t_a - \eta. \quad (7.69)$$

Moreover, there also exists a closed f -invariant subset Θ which is internally chain transitive but not topologically transitive such that

$$C_{\Theta} \subsetneq \Theta \text{ and } t_a^{\Theta} > t_a - \eta. \quad (7.70)$$

Proof. Since $M(f, X)$ is compact and $\mu \mapsto \int \varphi d\mu$ is continuous, there exist $\mu_{\max}, \mu_{\min} \in M(f, X)$ with

$$\int \varphi d\mu_{\max} = \sup_{\mu \in M(f, X)} \int \varphi d\mu \text{ and } \int \varphi d\mu_{\min} = \inf_{\mu \in M(f, X)} \int \varphi d\mu.$$

For any $\eta > 0$, by the definition of t_a , there exists a $\lambda \in M(f, X)$ with $\int \varphi d\lambda = a$ such that $h_{\lambda} > t_a - \eta/2$. Now choose $\theta \in (0, 1)$ close to 1 such that $\nu_1 := \theta\lambda + (1 - \theta)\mu_{\min}$ and $\nu_2 = \theta\lambda + (1 - \theta)\mu_{\max}$ satisfy that

$$h_{\nu_1} = \theta h_{\lambda} + (1 - \theta)h_{\mu_{\min}} \geq \theta h_{\lambda} > t_a - \eta/2 \text{ and } h_{\nu_2} = \theta h_{\lambda} + (1 - \theta)h_{\mu_{\max}} \geq \theta h_{\lambda} > t_a - \eta/2.$$

Then we let $\varepsilon := \min\{a - \int \varphi d\nu_1, \int \varphi d\nu_2 - a\} > 0$ and choose $\zeta > 0$ such that

$$\rho(\tau, \kappa) < \zeta \Rightarrow \left| \int \varphi d\tau - \int \varphi d\kappa \right| < \varepsilon/2.$$

By lemma B, there are three topologically expanding basic set $\Lambda_i \subseteq \Lambda \subsetneq X$ ($i = 1, 2$) with ergodic measures $\omega_i \in M(f, \Lambda_i)$, $i = 1, 2$ such that $h_{\omega_i} > h_{\nu_i} - \eta/2$. Moreover, $d_H(\nu_i, M(f, \Lambda_i)) < \zeta$. This implies that for any $\lambda_i \in M(f, \Lambda_i)$, one has $|\int \varphi d\lambda_i - \int \varphi d\nu_i| < \varepsilon/2$. So we have

$$\sup_{\mu \in M_f(\Lambda_1)} \int \varphi(x) d\mu \leq \int \varphi d\nu_1 + \varepsilon/2 \leq a - \varepsilon/2 \text{ and } \inf_{\mu \in M_f(\Lambda_2)} \int \varphi(x) d\mu \geq \int \varphi d\nu_2 - \varepsilon/2 \geq a + \varepsilon/2,$$

proving that $\sup_{\mu \in M_f(\Lambda_1)} \int \varphi(x) d\mu < a < \inf_{\mu \in M_f(\Lambda_2)} \int \varphi(x) d\mu$.

Finally, $h_{\text{top}}(\Lambda) \geq h_{\omega_i} > h_{\nu_i} - \eta/2 > t_a - \eta, i = 1, 2$.

Now by variational principle, we choose two ergodic measures $\nu_i \in M(f, \Lambda_i), i = 1, 2$ such that

$$h_{\nu_i} > t_a - \eta.$$

Due to (7.68), there exists a $0 < \theta < 1$ such that $\nu = \theta\nu_1 + (1 - \theta)\nu_2$ satisfies that

$$\int \varphi d\nu = \theta \int \varphi d\nu_1 + (1 - \theta) \int \varphi d\nu_2 = a.$$

Moreover, one has

$$h_\nu = \theta h_{\nu_1} + (1 - \theta) h_{\nu_2} \geq \min\{h_{\nu_1}, h_{\nu_2}\} > t_a - \eta.$$

This proves that $t_a^\Lambda > t_a - \eta$. Meanwhile, note that the periodic points are dense in Λ , there is a $\nu \in M(f, \Lambda)$ with full support [25]. Hence, $C_\Lambda = \Lambda$. Thus (7.69) is proved.

Furthermore, choose an arbitrary point $z \notin \Lambda$ such that $\omega_f(z) = \Lambda$. Let $A = \text{orb}(z, f) \cup \Lambda$. Note that A is closed and f -invariant. So by lemma 2.11, there is a point $x \in X$ such that

$$A \subseteq \omega_f(x) \subseteq \bigcup_{l=0}^{\infty} f^{-l}A.$$

In particular,

$$\overline{\bigcup_{y \in \omega_f(x)} \omega_f(y)} \subseteq A \subseteq \omega_f(x) \neq X.$$

Let $\Theta = \omega_f(x)$. Then Θ is chain transitive. By lemma 2.5,

$$M(f, \Theta) = M(f, A) = M(f, \Lambda).$$

This implies that $t_a^\Theta = t_a^\Lambda > t_a - \eta$. Moreover, note that $C_\Theta = \Lambda \subsetneq \Theta$. Thus (7.70) is proved. The proof of the fact that Θ is not topologically transitive follows a similar discussion as in (1). \square

7.2 Proof of Theorem A

Now let us prove theorem A item by item.

(1) For any $\eta > 0$, choose Λ satisfying (7.66). Then by proposition A, 5.12 and 6.20, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda).$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_\Lambda = \Lambda$. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Lambda) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

(1') For any $\eta > 0$, choose Θ satisfying (7.67). Then by proposition A, 5.12 and 6.20, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Theta \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap I_\varphi^\Theta(f)) = h_{\text{top}}(f|_\Theta).$$

Note that $C_\Theta \subsetneq \Theta$ and Θ is not topologically transitive. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Theta) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

(2) For any $\eta > 0$, choose Λ satisfying (7.66). Then by proposition A, 5.12 and 6.21, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda}).$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_{\Lambda} = \Lambda$. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Lambda}) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

(2') For any $\eta > 0$, choose Θ satisfying (7.67). Then by proposition A, 5.12 and 6.21, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Theta} \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap I_{\varphi}^{\Theta}(f)) = h_{\text{top}}(f|_{\Theta}).$$

Note that $C_{\Theta} \subsetneq \Theta$ and Θ is not topologically transitive. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Theta}) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

(3) For any $\eta > 0$, choose Λ satisfying (7.66). Then for any $Z \in \Xi_{\Lambda} \cup \{\emptyset\}$, proposition A, 5.12 and 6.22 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_{\varphi}^{\Lambda}(f)) = h_{\text{top}}(f|_{\Lambda}).$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_{\Lambda} = \Lambda$. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Lambda}) > h_{\text{top}}(f) - \eta.$$

Meanwhile, if we let $Z \in \Xi_{\Lambda}$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Lambda}) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

(3') For any $\eta > 0$, choose Θ satisfying (7.67). Then for any $Z \in \Xi_{\Theta} \cup \{\emptyset\}$, proposition A, 5.12 and 6.22 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Theta} \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap I_{\varphi}^{\Theta}(f)) = h_{\text{top}}(f|_{\Theta}).$$

Note that $C_{\Theta} \subsetneq \Theta$ and Θ is not topologically transitive. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Theta}) > h_{\text{top}}(f) - \eta.$$

Meanwhile, if we let $Z \in \Xi_{\Theta}$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_{\varphi}(f)) \geq h_{\text{top}}(f|_{\Theta}) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

- (4) For any $\eta > 0$, choose Λ satisfying (7.66). Then for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$, proposition A, 5.12 and 6.23 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap I_\varphi^\Lambda(f)) = h_{\text{top}}(f|_\Lambda).$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_\Lambda = \Lambda$. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Lambda) > h_{\text{top}}(f) - \eta.$$

Meanwhile, if we let $Z \in \Xi_\Lambda$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Lambda) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

- (4') For any $\eta > 0$, choose Θ satisfying (7.67). Then for any $Z \in \Xi_\Theta \cup \{\emptyset\}$, proposition A, 5.12 and 6.23 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Theta \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap I_\varphi^\Theta(f)) = h_{\text{top}}(f|_\Theta).$$

Note that $C_\Theta \subsetneq \Theta$ and Θ is not topologically transitive. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Theta) > h_{\text{top}}(f) - \eta.$$

Meanwhile, if we let $Z \in \Xi_\Theta$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap I_\varphi(f)) \geq h_{\text{top}}(f|_\Theta) > h_{\text{top}}(f) - \eta.$$

By the arbitrariness of η , one obtains the result.

- (5) For any $\eta > 0$, choose an ergodic measure ν such that $h_\nu > h_{\text{top}}(f) - \eta/2$. Then we use corollary 4.12 to find a minimal subset Λ_η such that Λ_η supports exactly two ergodic measures ν_1, ν_2 with $h_{\nu_1} > h_\nu - \eta/2 > h_{\text{top}}(f) - \eta$ and

$$\int \varphi d\nu^1 < \int \varphi d\nu^2.$$

Now choose $0 < \theta_1 < \theta_2 < 1$ close to 1 such that $\mu^i = \theta_i \nu^1 + (1 - \theta_i) \nu^2, i = 1, 2$ satisfy that

$$h_{\mu^i} \geq \theta_i h_{\nu^1} > h_\nu - \eta/2 > h_{\text{top}}(f) - \eta.$$

Let $K := \text{cov}\{\mu^1, \mu^2\}$. Then it is not hard to see that

$$G_K^{\Lambda_\eta, N} \subseteq QAP(f) \cap NR(f) \cap I_\varphi(f).$$

Moreover, proposition A and 5.12 indicate that (Λ_η, f) is nonrecurrently-star-saturated. So

$$h_{\text{top}}(G_K^{\Lambda_\eta, N}) = \inf\{h_{\mu^1}, h_{\mu^2}\} > h_{\text{top}}(f) - \eta.$$

Since η is arbitrary, we see that

$$h_{\text{top}}(QAP(f) \cap NR(f) \cap I_\varphi(f)) = h_{\text{top}}(f).$$

- (5') For any $\eta > 0$, choose Λ_η as obtained in the proof of (5) above. Since (X, f) is topologically expanding and transitive, we can find a $z \notin \Lambda_\eta$ such that $\omega_f(z) = \Lambda_\eta$. Let $A = \text{orb}(z, f) \cup \Lambda_\eta$. Note that A is closed and f -invariant. So by lemma 2.11, there is a point $x \in X$ such that

$$A \subseteq \omega_f(x) \subseteq \bigcup_{l=0}^{\infty} f^{-l}A.$$

In particular,

$$\overline{\bigcup_{y \in \omega_f(x)} \omega_f(y)} \subseteq A \subseteq \omega_f(x) \neq X.$$

Let $\Theta_\eta = \omega_f(x)$. Then Θ_η is chain transitive [37]. By lemma 2.5,

$$M(f, \Theta_\eta) = M(f, A) = M(f, \Lambda_\eta).$$

This implies that $C_{\Theta_\eta} = \Lambda_\eta \subsetneq \Theta_\eta$. Moreover, it is not hard to check that

$$G_K^{\Theta_\eta, N} \subseteq (WQAP(f) \setminus QAP(f)) \cap NR(f) \cap I_\varphi(f),$$

where K is defined in (5). In addition, proposition A and 5.12 indicate that (Θ_η, f) is nonrecurrently-star-saturated. So

$$h_{\text{top}}(G_K^{\Theta_\eta, N}) = \inf\{h_{\mu^1}, h_{\mu^2}\} > h_{\text{top}}(f) - \eta.$$

Since η is arbitrary, we see that

$$h_{\text{top}}(QAP(f) \cap NR(f) \cap I_\varphi(f)) = h_{\text{top}}(f).$$

□

7.3 Proof of Theorem B

- (1) For any $\eta > 0$, choose Λ satisfying (7.69). Then by proposition A, 5.12 and 6.20, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda.$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_\Lambda = \Lambda$. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Lambda > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

- (1') For any $\eta > 0$, choose Θ satisfying (7.70). Then by proposition A, 5.12 and 6.20, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) = C_\Theta \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap R_\varphi^\Theta(a)) = t_a^\Theta.$$

Note that $C_\Theta \subsetneq \Theta$ and Θ is not topologically transitive. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Theta > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap NR(f) \cap R_\varphi(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(2) For any $\eta > 0$, choose Λ satisfying (7.69). Then by proposition A, 5.12 and 6.21, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_{\varphi}^{\Lambda}(a)) = t_a^{\Lambda}.$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_{\Lambda} = \Lambda$. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a^{\Lambda} > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(2') For any $\eta > 0$, choose Θ satisfying (7.70). Then by proposition A, 5.12 and 6.21, one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_{\Theta} \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap R_{\varphi}^{\Theta}(a)) = t_a^{\Theta}.$$

Note that $C_{\Theta} \subsetneq \Theta$ and Θ is not topologically transitive. So one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_{\varphi}(a)) \geq t_a^{\Theta} > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid X_{\underline{d}}(x) = X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_{\varphi}(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(3) For any $\eta > 0$, choose Λ satisfying (7.69). Then for any $Z \in \Xi_{\Lambda} \cup \{\emptyset\}$, proposition A, 5.12 and 6.22 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_{\Lambda} \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_{\varphi}^{\Lambda}(a)) = t_a^{\Lambda}.$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_{\Lambda} = \Lambda$. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a^{\Lambda} > t_a - \eta.$$

Meanwhile, if we let $Z \in \Xi_{\Lambda}$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a^{\Lambda} > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a$$

and

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_{\varphi}(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(3') For any $\eta > 0$, choose Θ satisfying (7.69). Then for any $Z \in \Xi_\Theta \cup \{\emptyset\}$, proposition A, 5.12 and 6.22 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) = C_\Theta \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap R_\varphi^\Theta(a)) = t_a^\Theta.$$

Note that $C_\Theta \subsetneq \Theta$ and Θ is not topologically transitive. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Theta > t_a - \eta.$$

Meanwhile, if we let $Z \in \Xi_\Theta$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Theta > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a$$

and

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) = X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(4) For any $\eta > 0$, choose Λ satisfying (7.69). Then for any $Z \in \Xi_\Lambda \cup \{\emptyset\}$, proposition A, 5.12 and 6.23 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Lambda \subseteq \omega_f(x) = \Lambda\} \cap NR(f) \cap R_\varphi^\Lambda(a)) = t_a^\Lambda.$$

Note that Λ is transitive and satisfies the shadowing property. So $\{x \in X : \omega_f(x) = \Lambda\} \subset \Upsilon$. Note also that $C_\Lambda = \Lambda$. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Lambda > t_a - \eta.$$

Meanwhile, if we let $Z \in \Xi_\Lambda$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Lambda > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a$$

and

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = \omega_f(x)\} \cap \Upsilon \cap NR(f) \cap R_\varphi(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(4') For any $\eta > 0$, choose Θ satisfying (7.69). Then for any $Z \in \Xi_\Theta \cup \{\emptyset\}$, proposition A, 5.12 and 6.23 yields that

$$h_{\text{top}}(\{x \in X \mid Z = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) = C_\Theta \subseteq \omega_f(x) = \Theta\} \cap NR(f) \cap R_\varphi^\Theta(a)) = t_a^\Theta.$$

Note that $C_\Theta \subsetneq \Theta$ and Θ is not topologically transitive. So if we let $Z = \emptyset$, one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Theta > t_a - \eta.$$

Meanwhile, if we let $Z \in \Xi_\Theta$, then one has

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a^\Theta > t_a - \eta.$$

By the arbitrariness of η , one has

$$h_{\text{top}}(\{x \in X \mid \emptyset = X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a$$

and

$$h_{\text{top}}(\{x \in X \mid \emptyset \neq X_{\underline{d}}(x) \subsetneq X_{\overline{d}}(x) \subsetneq X_{B^*}(x) \subsetneq \omega_f(x)\} \cap \Upsilon_T^c \cap NR(f) \cap R_\varphi(a)) \geq t_a.$$

The opposite inequality follows from corollary 6.7.

(5) Denote

$$t_a^k := \sup\{h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X), S_\mu \neq X, S_\mu \text{ is minimal and } \#M(f, S_\mu) = k\}.$$

For any $\eta > 0$ and any $k \in \mathbb{N}$, choose a $\nu \in M(f, X)$ such that $S_\nu \neq X$, S_ν is minimal, $\#M(f, S_\nu) = k$ and moreover,

$$h_\nu > t_a^k - \eta.$$

Now let $K := \{\nu\}$. Then it is not hard to see that

$$G_K^{S_\nu, N} \subseteq QAP(f) \cap NR(f) \cap R_\varphi(a).$$

Furthermore, proposition A and 5.12 indicate that (S_ν, f) is nonrecurrently-star-saturated. So

$$h_{\text{top}}(G_K^{S_\nu, N}) = h_\nu > t_a^k - \eta.$$

Since η is arbitrary, we see that

$$h_{\text{top}}(QAP(f) \cap NR(f) \cap R_\varphi(a)) \geq t_a^k.$$

Meanwhile, it is clear that

$$\begin{aligned} h_{\text{top}}(R_\varphi(a)) &\geq h_{\text{top}}(R_\varphi(a) \cap NR(f)) \\ &\geq h_{\text{top}}(R_\varphi(a) \cap NR(f) \cap WQAP(f)) \\ &\geq h_{\text{top}}(R_\varphi(a) \cap NR(f) \cap QAP(f)) \end{aligned}$$

and

$$\sup\{h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X)\} \geq \sup\{h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X) \text{ and } S_\mu \neq X\} \geq t_a^k.$$

Note also that

$$h_{\text{top}}(R_\varphi(a)) = \sup\{h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X)\}.$$

So it is left to show that

$$t_a^k \geq \sup\{h_\mu \mid \int \varphi d\mu = a, \mu \in M(f, X)\}. \quad (7.71)$$

Indeed, for any $\eta > 0$, choose some $\mu \in M(f, X)$ such that $h_\mu > t_a - \eta$. Then choose $\gamma > 0$ small enough such that $k\gamma < L_2 - a - \gamma/2$ and

$$\frac{\frac{1}{2}(L_2 - a)}{\frac{1}{2}(L_2 - a) + \gamma} h_\mu > t_a - \eta. \quad (7.72)$$

According to lemma C, there is a minimal subset \overline{X} satisfies properties 1-3 there. By property 2, we choose $0 < \theta < 1$ such that $\nu = \theta\nu_1 + (1 - \theta)\nu_2 \in M(f, \overline{X})$ satisfies that

$$\int \varphi d\mu = \theta \int \varphi d\nu_1 + (1 - \theta) \int \varphi d\nu_2 = a.$$

Note that $S_\nu = S_{\nu_1} \cup S_{\nu_2} = \overline{X} \neq X$ is minimal and $\#M(f, \overline{X}) = k$. Moreover,

$$\theta = \frac{\int \varphi d\nu_2 - a}{\int \varphi d\nu_2 - \int \varphi d\nu_1} > \frac{\frac{1}{2}(L_2 - a)}{\frac{1}{2}(L_2 - a) + a - \int \varphi d\nu_1} > \frac{\frac{1}{2}(L_2 - a)}{\frac{1}{2}(L_2 - a) + \gamma}.$$

Then (7.72) produces that

$$h_\nu > \theta h_\mu > t_a - \eta.$$

Since η is arbitrary, we complete our proof. □

8 Non-expansive case

We aim to characterize the almost periodic points and non-recurrent points in the general case here.

Theorem 8.1. Suppose (X, f) satisfies the shadowing property or the almost specification. Then

(1) $h_{\text{top}}(AP(f)) = h_{\text{top}}(f).$

(2) $h_{\text{top}}(NR(f)) = h_{\text{top}}(f).$

In particular, we have a conclusion for generic dynamical systems. Let M be a compact manifold with a *decomposition*. We can induce normalized Lebesgue measure \mathcal{L} on M , projecting it from D^n . For every $1 \leq i \leq k$ take homeomorphisms $\xi_i : H_i \rightarrow D^n$ and denote by m the Lebesgue measure on \mathbb{R}^n . Then the sets $A \subseteq M$ such that $\xi_i(A \cap H_i)$ is a Lebesgue measurable set in \mathbb{R}^n are called *Lebesgue measurable on M* . Clearly, it is a well defined σ -algebra. For any (Lebesgue) measurable set $A \subseteq M$, we define

$$\mathcal{L}(A) := \frac{1}{k} \sum_{i=1}^k \frac{m(\xi_i(A \cap H_i))}{m(D^n)}. \quad (8.73)$$

One sees that \mathcal{L} is non-atomic and positive on open sets since ξ_i is homeomorphism. Moreover, $\mathcal{L}(M) = 1$. Thus \mathcal{L} is well defined lebesgue measure on M . Let $C(M)$ be the set of continuous maps on M and $H(M)$ the set of homeomorphisms on M . We endow $C(M)$ with the metric

$$d_C(f, g) = \sup_{x \in M} d(fx, gx)$$

and $H(M)$ with the metric

$$d_H(f, g) = d_C(f, g) + d_C(f^{-1}, g^{-1}).$$

Both spaces $(C(M), d_C)$, $(H(M), d_H)$ are complete. A subset \mathcal{R} of a metric space X is *residual* if it contains a countable intersection of dense open sets. Recall that residual subsets of complete spaces are always dense by Baire theorem.

Corollary 8.2. Let M be a compact topological manifold (with or without boundary) of dimension at least 2 and assume that M admits a decomposition. Then there is a residual subset $\mathcal{R} \subseteq H(M)$ (or $\mathcal{R} \subseteq C(M)$) such that for any $f \in \mathcal{R}$,

$$(1) \ h_{\text{top}}(AP(f)) = h_{\text{top}}(f).$$

$$(2) \ h_{\text{top}}(NR(f)) = h_{\text{top}}(f).$$

Proof. The result follows from the fact that C^0 generic $f \in H(M)$ (or $f \in C(M)$) has the shadowing property (see [42] and [40, 41], respectively) and theorem 8.1 above. \square

8.1 Construction of Symbolic Factor Subsystem

Recall the following characterization for systems with the shadowing property.

Lemma 8.3. [26] Suppose that (X, f) has the shadowing property and $h_{\text{top}}(f) > 0$. Then for any $0 < \alpha < h_{\text{top}}(f)$ there are $m, k \in \mathbb{N}$, $\log(m)/k > \alpha$ and a closed set $\Lambda \subseteq X$ invariant under f^k such that there is a factor map $\pi: (\Lambda, f^k) \rightarrow (\Sigma_{m+1}^+, \sigma)$. If in addition, (X, f) is positively expansive, then π is a conjugation.

A similar discussion leads to the following result.

Lemma 8.4. Suppose (X, f) satisfies the almost specification property. Then for any $0 < \alpha < h_{\text{top}}(f)$, there are $m, k \in \mathbb{N}$, $\log(m)/k > \alpha$ and a closed set $\Lambda \subseteq X$ invariant under f^k such that there is a factor map $\pi: (\Lambda, f^k) \rightarrow (\Sigma_m^+, \sigma)$.

Proof. By the variational principle, choose some invariant measure ν such that $0 < \alpha < h_\nu \leq h_{\text{top}}(f)$. According to lemma 2.19, there exist $\delta > 0$ and $\varepsilon > 0$ so that for each neighborhood F of ν in $M(X)$, there exists $n_F \in \mathbb{N}$ such that for any $n \geq n_F$, there exists $\Gamma_n \subseteq X_{n,F}$ which is (δ, n, ε) -separated and satisfies $\log |\Gamma_n| \geq n\alpha$. Now choose an arbitrary neighbourhood F of ν . Since $\lim_{n \rightarrow \infty} \frac{g(n, \varepsilon/3)}{n} = 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, $g(n, \frac{\varepsilon}{3}) \leq \frac{\delta}{3}$. Let $k = \max\{n_F, k_g(\frac{\varepsilon}{3}), [\frac{3}{\delta}] + 1\}$. Enumerate the elements of Γ_k as $\{p_1, \dots, p_m\}$ where $m = |\Gamma_k|$.

Let Σ_m^+ be the set whose element is $(a_0 a_1 \dots a_n \dots)$ such that $a_i \in \{p_1, \dots, p_m\}$, $i \in \mathbb{Z}$. For every $\xi \in \Sigma_m$, denote

$$Y_\xi = \left\{ z \in X : f^{ik}(z) \in B_k(g; \xi_i, \frac{\varepsilon}{3}) \text{ for } i \in \mathbb{Z} \right\}.$$

By the almost specification property, the selection of k and the compactness of X , Y_ξ is nonempty. Note that if $\xi \neq \psi$ then there is t such that $\xi_t \neq \psi_t$. For any $x \in Y_\xi$ and $y \in Y_\psi$, since $\delta - 2g(k, \frac{\varepsilon}{3}) \geq \frac{\delta}{3}$, $f^{tk}x$ and $f^{tk}y$ are $(\frac{\delta}{3}, k, \frac{\varepsilon}{3})$ -separated. Moreover, $\frac{\delta}{3} \cdot n \geq 1$. So x, y are $(tk, \frac{\varepsilon}{3})$ -separated which implies that $x \neq y$. Therefore, $Y_\xi \cap Y_\psi = \emptyset$. So we define Λ as the disjoint union of Y_ξ :

$$\Lambda = \bigsqcup_{\xi \in \Sigma_r^+} Y_\xi.$$

Note that $f^k(Y_\xi) \subseteq Y_{\sigma(\xi)}$. So Λ is f^k -invariant. It is not hard to see that if $x \in Y_\xi$ and Y_ψ and $d(f^l(x), f^l(y)) < \varepsilon/3$ for $l = 0, \dots, ks - 1$ then $\xi_i = \psi_i$ for $i = 0, \dots, s - 1$. Therefore, if we define $\pi: \Lambda \rightarrow \Sigma_r^+$ as

$$\pi(x) := \xi \text{ if } x \in Y_\xi,$$

then π is a continuous surjection. This shows that Λ is closed and clearly also $\sigma \circ \pi = \pi \circ f^k$.

Finally, observe that

$$\frac{\log(m)}{k} = \frac{\log |\Gamma_k|}{k} > \alpha.$$

The proof is completed. \square

8.2 Proof of Theorem 8.1

When $h_{\text{top}}(f) = 0$, there is nothing to prove. So we suppose $h_{\text{top}}(f) > 0$. By lemma 8.3 and 8.4, for any $0 < \alpha < h_{\text{top}}(f)$, there are $m, k \in \mathbb{N}$, $\log(m)/k > \alpha$ and a closed and f^k -invariant set $\Lambda \subseteq X$ with a semiconjugation $\pi : (\Lambda, f^k) \rightarrow (\Sigma_m^+, \sigma)$.

(1) For (Σ_m^+, σ) , there is a minimal subsystem $(\Sigma, \sigma|_\Sigma)$ such that $h_{\text{top}}(\Sigma) > k\alpha$ [35]. Now $\pi^{-1}(\Sigma)$ is a closed and f^k -invariant set of X . By Zorn lemma, there exists a f^k -minimal set $\Delta \subseteq \pi^{-1}(\Sigma)$. Then $\pi(\Delta)$ is $\sigma|_\Sigma$ -invariant and minimal. Since $\pi(\Delta) \subseteq \Sigma$ and Σ is minimal, $\pi(\Delta) = \Sigma$. Then $\pi|_\Delta : (\Delta, f^k) \rightarrow (\Sigma_m^+, \sigma)$ is a semiconjugation so that $h_{\text{top}}(f^k, \Delta) \geq h_{\text{top}}(\sigma, \Sigma) \geq k\alpha$. Whereby $h_{\text{top}}(f, \Delta) = \frac{1}{k}h_{\text{top}}(f^k, \Delta) \geq \alpha$. Meanwhile, $AP(f^k) = AP(f)$.

So $\Delta \subseteq AP(f)$. Since α is arbitrary, we are done.

(2) Since (Σ_m^+, σ) is topologically expanding and transitive, by Theorem A, we obtain that $h_{\text{top}}(NR(\sigma)) = h_{\text{top}}(\sigma, \Sigma_m^+) > k\alpha$. However, by (2.2), $\pi(Rec(f^k)) = Rec(\sigma)$, so $NR(f^k, \Delta) \supseteq NR(\sigma)$. Meanwhile, note that $NR(f^k) = NR(f)$. Therefore, since π is a semiconjugation,

$$h_{\text{top}}(f, NR(f)) = \frac{1}{k}h_{\text{top}}(f^k, NR(f)) = \frac{1}{k}h_{\text{top}}(f^k, NR(f^k)) \geq \frac{1}{k}h_{\text{top}}(\sigma, NR(\sigma)) > \alpha.$$

By the arbitrariness of α , we see $h_{\text{top}}(NR(f)) = h_{\text{top}}(f)$. □

□

9 Proof of Proposition 4.3

9.1 Notions and Notations

Fix any positive integer $k \geq 2$ and consider the following set $\Sigma_k^+ = \{0, 1, \dots, k-1\}^{\mathbb{N}_0}$ with the product topology induced by the discrete topology on $\{0, 1, \dots, k-1\}$. The space Σ_k^+ is always endowed with the *shift map* σ defined by $\sigma(x)_i = x_{i+1}$ for every integer $i \geq 0$. It is not hard to verify that σ is continuous and Σ_k^+ is a compact metrizable space. We endow it with the (compatible) metric defined by $d(x, y) = 2^{-k}$ when $x \neq y$ and $k = \min\{i : x_i \neq y_i\}$. Dynamical system (Σ_k^+, σ) or simply Σ_k^+ for short, is called *full shift*. By *subshift* or *shift* we mean any compact and σ -invariant subset of Σ_k^+ .

A finite sequence (a_1, \dots, a_N) of elements $a_j \in S$ is called a *block* (or *N-block*) in Σ_k^+ . N is called the length of the block. The block $A = (a_1, \dots, a_N)$ is said to occur in $x \in \Sigma_k^+$ at the place of m if $x_m = a_1, \dots, x_{m+N-1} = a_N$. In this case one writes $A \prec x$. The block A is said to occur in the subshift $\Lambda \subseteq \Sigma_k^+$ if there exists an $x \in \Lambda$ such that $A \prec x$. In this case, one writes $A \prec \Lambda$.

For any $m \in \mathbb{Z}$ and any block $A = (a_1, \dots, a_N)$ in Σ_k^+ . Let ${}_m[a_1, \dots, a_N]$ denote the set of all $x \in \Sigma_k^+$ such that (a_1, \dots, a_N) occurs in x at the place m . The set ${}_m[a_1, \dots, a_N]$ is called a *cylinder of length N* based on the block (a_1, \dots, a_N) at the place m .

It follows easily from the product topology that a cylinder is both open and closed. Then it is not hard to show that $\text{span}\{1_P : P \in Bl(X)\}$ is dense in $C(X)$ via Stone-Weierstrass theorem. Here 1_P denotes the characteristic function of P and $\text{span } A$ denotes the linear space generated by A . So we can define a metric on Σ_k^+ for the *weak** topology as follows.

$$\rho(\mu, \nu) = \sum_{\substack{r \geq 1 \\ P \in Bl_r(\Omega_k)}} \frac{|\mu(P) - \nu(P)|}{2^r \cdot k^r}.$$

Therefore, we have the following characterization.

Lemma 9.1. Then for any $\varepsilon > 0$, there exist $t = t(\varepsilon) > 0$ and $\delta = \delta(t, \varepsilon) > 0$ such that

$$\mu, \nu \in M(\Sigma_k^+), |\mu(P) - \nu(P)| < \delta \forall P \in Bl_t(\Sigma_k^+) \Rightarrow \rho(\mu, \nu) < \varepsilon.$$

We denote the set of all blocks by Bl and the set of blocks of length r by Bl_r . For $P = (p_1, \dots, p_m) \in Bl_m, Q = (q_1, \dots, q_n) \in Bl_n$, we write $PQ = P \cdot Q = (p_1, \dots, p_m, q_1, \dots, q_n) \in Bl_{m+n}$, the juxtaposition of P and Q . If $l(Q) \geq l(P)$, we define

$$\mu_Q(P) = (l(Q) - l(P) + 1)^{-1} \cdot \text{card}\{j : 1 \leq j \leq l(Q) - l(P) + 1, (q_j, \dots, q_{j+l(P)-1}) = P\}$$

as the relative frequency of (the occurrence of) P in Q as a subblock. Then μ_Q is a probability measure on Bl_m for every $m \leq l(Q)$. If Q occurs in x at the place m , then one sees that

$$\mu_Q(P) = \langle 1_P, \mathcal{E}_{l(Q)-l(P)+1}(\sigma^m(x)) \rangle. \quad (9.74)$$

For $w \in \Sigma_k^+, O_w = \{\sigma^j w \mid j \in \mathbb{Z}\}$ is the orbit of w and if $s \leq t \in \mathbb{Z}$:

$$w_{\langle s, t \rangle} = (w_s, \dots, w_t) \in Bl_{t-s+1}.$$

For a subshift K of Σ_k^+ and $r \in \mathbb{N}$,

$$Bl_r(K) = \{P \in Bl_r \mid {}_0[P] \cap K \neq \emptyset\} = \{w_{\langle 1, r \rangle} \mid w \in K\}$$

is the set of r -blocks occurring in K , and $Bl(K) = \bigcup_{r \in \mathbb{N}} Bl_r(K)$ is the set of all K -blocks. For the number of blocks we write

$$\theta_r(K) = \text{card } Bl_r(K), \theta(K) = \theta_1(K).$$

We shall often assume that we have some subshift K without specifying Σ_k^+ . In such a case we use $\theta(K)$ as a bound for the number of states.

We write the topological entropy of K as $h(K) = h_{\text{top}}(\sigma|_K)$. We have [25, Proposition 16.11]:

$$h(K) = \lim_{r \rightarrow \infty} r^{-1} \log \theta_r(K). \quad (9.75)$$

The symbol \prec will express all kinds of occurrence of blocks:

a) If $P, Q \in Bl(M)$, then $P \prec Q$ if P is a subblock of Q , or equivalently, $\mu_Q(P) > 0$. If $P \prec Bl$, $\omega \in \Sigma_k^+$, then $P \prec \omega$ if P occurs in ω or equivalently, $\omega \in \bigcup_{i=-\infty}^{+\infty} {}_i[P]$. If $P \in Bl(M)$ and $K \subseteq \Sigma_k^+$ is subshift, then $P \prec K$ if there is an $\omega \in K$ with $P \prec \omega$, or equivalently, $P \in Bl(K)$.

b) If $P \in Bl(M)$ and $\omega \in \Sigma_k^+$ (resp. $K \subseteq \Sigma_k^+$ is a subshift), then $P \prec_d \omega$ (resp. $P \prec_d K$) if there is a $k \in \mathbb{N}$ such that the distances of the occurrences of P in ω (resp. in $\eta \in K$) are at most k , i.e.

$$O(\omega) \subseteq \bigcup_{i=0}^{k-1} {}_i[P] \text{ (resp. } K \subseteq \bigcup_{i=0}^{k-1} {}_i[P]).$$

Because of the compactness of K , $P \prec_d K \Leftrightarrow \forall \eta \in K : P \prec_d \eta$. We say that P “occurs densely”.

c) If $P \in Bl(M)$, $\omega \in \Sigma_k^+$ (resp. $K \subseteq \Sigma_k^+$ is a subshift) and $\varepsilon > 0$, then $P \prec_{\varepsilon\text{-reg}} \omega$ (resp. $P \prec_{\varepsilon\text{-reg}} K$) if there is some $t > l(P)$ such that

$$Q_1, Q_2 \prec \omega \text{ (resp. } Q_1, Q_2 \prec K), l(Q_i) = t \Rightarrow |\mu_{Q_1}(P) - \mu_{Q_2}(P)| < \varepsilon.$$

Equivalently, $P \prec_{\varepsilon\text{-reg}} \omega$ (resp. $P \prec_{\varepsilon\text{-reg}} K$) if there is some $s > l(P)$ such that

$$Q_1, Q_2 \prec \omega \text{ (resp. } Q_1, Q_2 \prec K), l(Q_i) \geq s \Rightarrow |\mu_{Q_1}(P) - \mu_{Q_2}(P)| < \varepsilon.$$

We have the following lemma [25, Theorem 26.7].

Lemma 9.2. The subshift $K \subseteq \Sigma_k^+$ is minimal if and only if $\forall P \prec K : P \prec_d K$.

9.2 Subshifts of Finite Type

Let B be a set of blocks occurring in Σ_k^+ . The subshift defined by excluding B is the set

$$\Lambda_B := \{x \in \Sigma_k^+ \mid \text{no block } \beta \in B \text{ occurs in } x\}.$$

Clearly Λ_B is shift-invariant and closed for its complement is open.

A subshift M is called a *subshift of finite type* (f.t. subshift for short) if there exists a finite excluded block system for M . We say that $N \in \mathbb{N}$ is an order of the f.t. subshift M if there exists an excluded block system B for M , which only contains blocks of length $\leq N$.

If $P, Q \prec M$, a M -transition block from P to Q is a block $U \prec M$ such that $P \cdot U \cdot Q \prec M$. We say $L \in \mathbb{N}$ is a transition length from P to Q if there exist M -transition blocks $U \in Bl_L(M)$ and $U' \in Bl_{L+1}(M)$ from P to Q . Furthermore, we say $L_0 \in \mathbb{N}$ is a transition length for M if for all $L \geq L_0$ and $P, Q \prec M$, L is a transition length from P to Q .

We have several equivalent descriptions.

Lemma 9.3. [25] The following two conditions are equivalent.

- 1) M is a f.t. subshift of order N .
- 2) There exists a block system B' of blocks of length N such that

$$\Lambda = \{(x_n)_{n \in \mathbb{Z}} \mid \forall n \in \mathbb{Z} : (x_n, x_{n+1}, \dots, x_{n+N-1}) \in B'\}.$$

A set B' of blocks defining a subshift M in this way will be called a defining system of blocks for M .

Lemma 9.4. [25] The following conditions are equivalent:

1. (M, σ) is topologically transitive.
2. For every i, j occurring in M , there exists an $n \in \mathbb{N}$ such that $\{x \in M \mid x_0 = i, x_n = j\} \neq \emptyset$.

Lemma 9.5. [25] The following three conditions are equivalent:

- a) M is an m.f.t.-subshift;
- b) there exists a transition length;
- c) M is topologically transitive and there are two blocks $P, Q \prec M$ (both at least as long as the order of M) such that there exists a transition length from P to Q .

9.3 Basic Lemmas

Lemma 9.6. [25, Lemma 26.16] Let M be a m.f.t.-subshift, $K_1, K_2 \subseteq M$ disjoint subshifts with $K_1 \neq \emptyset$ and $\varepsilon > 0$.

- a) There is an m.f.t.-subshift M_1 with

$$K_1 \subseteq M_1 \subseteq M, \quad K_2 \cap M_1 = \emptyset \text{ and } h(M_1) < h(K_1) + \varepsilon.$$

- b) If C, C' are finite sets of M -blocks and K_1 has one or both of the following properties

$$P \in C \Rightarrow P \prec_d K_1; \quad P \in C' \Rightarrow P \prec_{\varepsilon-reg} K_1,$$

then the corresponding properties can be achieved for M_1 .

Lemma 9.7. Let M be a m.f.t.-subshift, $\nu \in \mathfrak{M}_\sigma(M)$ be ergodic and $0 < \widehat{h} < h_\nu$. Then for any $\widehat{t} \in \mathbb{N}$ and $\delta > 0$, there exists a subshift $\widehat{M} \subsetneq M$ and $\widehat{r} \in \mathbb{N}$ such that

- i) $|\mu_Q(P) - \nu(P)| < \delta$ for all $P \in \bigcup_{t \leq \widehat{t}} Bl_t(M)$ and $Q \in \bigcup_{r \geq \widehat{r}} Bl_r(\widehat{M})$.
- ii) $h(\widehat{M}) > \widehat{h}$.

Proof. Consider the following neighborhood G of ν :

$$G := \{\mu \in \mathfrak{M}(M) \mid |\mu(P) - \nu(P)| < \delta \text{ for } P \in \bigcup_{t \leq \widehat{t}} Bl_t(M)\}.$$

Since M has the basic-entropy-dense property, we obtain a subshift $\widetilde{M} \subseteq M$ with $h(\widetilde{M}) > \widehat{h}$ and $n_G \in \mathbb{N}$ such that $\mathcal{E}_n(x) \in G$ for all $x \in \widetilde{M}$ and $n \geq n_G$. This shows that $\widetilde{M} \neq M$. Then it is not hard to use (9.74) and lemma 9.6 to find the required $\widehat{M} \supseteq \widetilde{M}$ and \widehat{r} . \square

Furthermore, we have the following lemma.

Lemma 9.8. Let M be a m.f.t.-subshift, $\nu \in \mathfrak{M}_\sigma(M)$ be ergodic and $0 < \bar{h} < h_\nu$. Then for any $\bar{t} \in \mathbb{N}$ and $\delta > 0$, there exists a m.f.t.-subshift $\overline{M} \subseteq X$ and $\bar{r} \in \mathbb{N}$ such that

- 1) $P \in Bl_{\bar{t}}(M), Q \in Bl_{\bar{r}}(\overline{M}) \Rightarrow P \prec Q$.
- 2) $|\mu_Q(P) - \nu(P)| < \delta$ for all $P \in \bigcup_{t \leq \bar{t}} Bl_t(M)$ and $Q \in \bigcup_{r \geq \bar{r}} Bl_r(\overline{M})$.
- 3) $h(\overline{M}) > \bar{h}$.

Proof. By lemma 9.7, we obtain a subshift $\widehat{M} \subsetneq M$ and $\widehat{r} \in \mathbb{N}$ such that

- a) $|\mu_Q(P) - \nu(P)| < \delta/2$ for all $P \in \bigcup_{t \leq \bar{t}} Bl_t(M)$ and $Q \in \bigcup_{r \geq \widehat{r}} Bl_r(\widehat{M})$.
- b) $h(\widehat{M}) > \frac{\bar{h} + h_\nu}{2}$.

Since $\widehat{M} \neq M$, there exists an $s \in \mathbb{N}$ such that $Bl_s(\widehat{M}) \neq Bl_s(M)$. Now choose a block $A \in Bl_s(M) \setminus Bl_s(\widehat{M})$. Since M is topological mixing, there is a transition length L so large that for any $Q, Q' \in Bl(M)$, there is $U \in Bl_L(M)$ and $U' \in Bl_{L+1}(M)$ with $QUQ' \prec M$ and $QU'Q' \prec M$. Furthermore, U, U' can be chosen such that (by the topological mixing property)

$$P \in Bl_{\bar{t}}(M) \Rightarrow P \prec U \text{ and } P \prec U'. \quad (9.76)$$

Moreover,

$$A \prec U \text{ and } A \prec U' \quad (9.77)$$

so that U, U' are ‘recognizable’.

For sufficient large $k \in \mathbb{N}$ such that

$$1. \quad \frac{\ln \theta_m(\widehat{M})}{m} \geq h_{\text{top}}(\widehat{M}) \text{ when } m \geq k. \quad (9.78)$$

$$2. \quad \frac{L+1}{k+L+1} \leq \delta/2. \quad (9.79)$$

3.

$$\frac{k}{k+L+1} \cdot \frac{\bar{h} + h_\nu}{2} > \bar{h}. \quad (9.80)$$

Define the subshift \overline{M} having the following defining block systems:

$$\begin{aligned} Bl_{k+L+1}(\overline{M}) = & \{P \in Bl_{k+L+1}(M) : P \prec QUR, \text{ where } Q, R \text{ run through } Bl_k(\widehat{M}) \text{ and} \\ & U \in Bl_L(M) \cup Bl_{L+1}(M) \text{ is a } M \text{ transition block from } Q \text{ to } R \text{ satisfying (9.76) and (9.77)}\}. \end{aligned}$$

By proposition 9.3, 9.4 and 9.5, \overline{M} is a m.f.t. subshift. By (9.76), one obtains Property 1) of \overline{M} . Let $\bar{r} > \hat{r}$ be sufficient large. Moreover, one observes the fact every $k+L+1$ -block in \overline{M} carries a k -block in \widehat{M} . Then one uses the property a) of \widehat{M} and (9.79) to check property 2). Finally, let us calculate the topological entropy of \overline{M} to substantiate property 3) of \overline{M} . Indeed, for any $l \in \mathbb{N}$, there are blocks of the following form

$$Q_1 U_1 R_1 U'_1 Q_2 U_2 R_2 U'_2 \cdots Q_l U_l R_l$$

occurring in \overline{M} , where $Q_i, R_i \in Bl_k(\widehat{M})$ and $U_i, U'_i \in Bl_L(\widehat{M}) \cup Bl_{L+1}(\widehat{M})$ are transition blocks. So one infers by (9.78) and (9.80) that

$$h_{\text{top}}(\overline{M}) = \lim_{r \rightarrow \infty} \frac{\ln \theta_r(\overline{M})}{r} = \lim_{l \rightarrow \infty} \frac{\ln \theta_{2l(k+L+1)}(\overline{M})}{2l(k+L+1)} \geq \lim_{l \rightarrow \infty} \frac{2l\theta_k(\widehat{M})}{2l(k+L+1)} \geq \frac{k}{k+L+1} \cdot \frac{\bar{h} + h_\nu}{2} > \bar{h}.$$

□

9.4 Proof of proposition 4.3

For each $1 \leq j \leq k$, choose a strictly decreasing sequence $\{h_i^j\}_{i=0}^\infty$ such that

$$h_0^j = h_{\mu^j} \text{ and } \lim_{i \rightarrow \infty} h_i^j = h_{\mu^j} - \eta.$$

We now inductively construct $k+1$ sequences of m.f.t. subshifts $\{M_i\}_{i=0}^\infty$ and $\{M_i^j\}_{i=0}^\infty$, $j = 1, \dots, k$. Let $M_0 = M$. According to lemma 9.1, for each $i \in \mathbb{N}$, there exist $\bar{t}_{i-1} > 0$ and $\bar{\delta}_{i-1} > 0$ such that

$$\lambda, \tau \in \mathfrak{M}(M), |\lambda(P) - \tau(P)| < \bar{\delta}_{i-1} \forall P \in Bl_{\bar{t}_{i-1}}(M_0) \Rightarrow \rho(\lambda, \tau) < 3^{-i-1}\varepsilon. \quad (9.81)$$

Let $t_0 = \max\{\bar{t}_0, 1\}$ and $\delta_0 = \min\{\bar{\delta}_0, 1\}$. Then using lemma 9.8, we obtain k m.f.t. subshift $M_0^1, M_0^2, \dots, M_0^k$ and $\bar{\tau}_0 \in \mathbb{N}$ satisfying that

i) $|\lambda_0^j(P) - \mu^j(P)| < \delta_0$ for all $\lambda_0^j \in \mathfrak{M}_\sigma(M_0^j)$ and $P \in \bigcup_{t \leq t_0} Bl_t(M_0)$ ($j = 1, \dots, k$). Consequently,

$$\rho(\lambda_0^j, \mu^j) < 3^{-1}\varepsilon \text{ for all } \lambda_0^j \in \mathfrak{M}_\sigma(M_0^j). \quad (9.82)$$

ii) $h(M_0^j) > h_1^j$ ($j = 1, \dots, k$).

Therefore, (9.82) and the selection of ε in (4.12) indicate that

$$\rho(\lambda_0^s, \lambda_0^t) > 3^{-1}\varepsilon > 0 \text{ for any } \lambda_0^s \in \mathfrak{M}_\sigma(M_0^s) \text{ and } \lambda_0^t \in \mathfrak{M}_\sigma(M_0^t), 1 \leq s < t \leq k. \quad (9.83)$$

Consequently, $M_0^s \cap M_0^t = \emptyset$. For otherwise the nonempty intersection is a closed invariant set which supports some invariant measure belonging to both $\mathfrak{M}_\sigma(M_0^s)$ and $\mathfrak{M}_\sigma(M_0^t)$, a contradiction to (9.83).

Suppose in the $(i-1)$ th step of the construction we have obtained a m.f.t.-subshift M_{i-1} and disjoint m.f.t.-subshifts $\{M_{i-1}^j\}_{j=1}^k$ of M_{i-1} with

$$h(M_{i-1}^j) > h_{i-1}^j, j = 1, \dots, k. \quad (9.84)$$

Then there exist k ergodic measures $\nu_{i-1}^j \in \mathfrak{M}_\sigma(M_{i-1}^j) (j = 1, \dots, k)$ such that

$$h_{\nu_{i-1}^j} > h_{i-1}^j, \quad j = 1, \dots, k.$$

We now choose $t_{i-1} > \max\{\bar{t}_{i-1}, 2^{i-1}\}$ so large that

$$Bl_{t_{i-1}}(M_{i-1}^s) \cap Bl_{t_{i-1}}(M_{i-1}^t) = \emptyset, 1 \leq s < t \leq k$$

and

$$\log \theta_{t_{i-1}}(M_{i-1}^j) > t_{i-1} h_{i-1}^j, \quad j = 1, \dots, k. \quad (9.85)$$

Here (9.85) comes from (9.75). For h_i^j and $0 < \delta_{i-1} < \min\{\bar{\delta}_{i-1}, 2^{-i+1}\}$, a convenient use of lemma 9.8 produces k m.f.t. subshifts $\{M_i^j\}_{j=1}^k$ and $\bar{r}_i > t_{i-1}$ such that

i)

$$P \in Bl_{t_{i-1}}(M_{i-1}), Q \in Bl_{\bar{r}_i}(M_i^j) \Rightarrow P \prec Q (j = 1, \dots, k). \quad (9.86)$$

ii) $|\mu_Q(P) - \nu_{i-1}^j(P)| < \delta_{i-1}$ for all $P \in \bigcup_{t \leq t_{i-1}} Bl_t(M_{i-1}), Q \in \bigcup_{r \geq \bar{r}_i} Bl_r(M_i^j)$. Consequently,

$$|\lambda_i^j(P) - \nu_{i-1}^j(P)| < \delta_{i-1} \text{ for all } \lambda_i^j \in \mathfrak{M}_\sigma(M_i^j) \text{ and } P \in \bigcup_{t \leq t_{i-1}} Bl_t(M_{i-1}) (j = 1, \dots, k). \quad (9.87)$$

iii) $h(M_i^j) > h_i^j (j = 1, \dots, k)$.

Moreover, by lemma 6.4, $M_i^1 \cup \dots \cup M_i^k \neq M_{i-1}$. So \bar{r}_i can be chosen such that

$$Bl_{\bar{r}_i}(M_i^1 \cup M_i^2 \cup \dots \cup M_i^k) \neq Bl_{\bar{r}_i}(M_{i-1}). \quad (9.88)$$

Now choose a block $C_i \in Bl_{\bar{r}_i}(M_{i-1}) \setminus Bl_{\bar{r}_i}(M_i^1 \cup \dots \cup M_i^k)$ and a M_{i-1} transition length L_i which is so large that for $P^s \in Bl(M_{i-1}^s)$ and $P^t \in Bl(M_{i-1}^t)$, $1 \leq s \neq t \leq k$, there are blocks $U_i^{st} \in Bl_{L_i}(M_{i-1})$ with

$$P^s U_i^{st} P^t \prec M_{i-1}. \quad (9.89)$$

and

$$C_i \prec U_i^{st} \text{ but only once, namely at the place } [\frac{L_i}{2}]. \quad (9.90)$$

Choose an integer $r_i > 4(\bar{r}_i + L_i) \cdot \varepsilon_{i-1}^{-1}$ and bigger than the order of M_{i-1} . Then choose k fixed blocks $S_i^j \in Bl_{r_i}(M_i^j) (j = 1, \dots, k)$. Furthermore, choose $(n^2 - n)$ fixed transition blocks $U_i^{st} (1 \leq s \neq t \leq k)$ satisfying (9.89), (9.90) and such that

$$S_i^s U_i^{st} S_i^t \prec M_{i-1}, 1 \leq s \neq t \leq k.$$

Let M_i is the m.f.t.-subshift having

$$Bl_{r_i}(M_i) = Bl_{r_i}(M_i^1 \cup \dots \cup M_i^k) \bigcup \{Q \in Bl_{r_i}(M_{i-1}) \mid Q \prec S_i^s U_i^{st} S_i^t, 1 \leq s \neq t \leq k\}$$

as defining block system. By proposition 9.3, 9.4 and 9.5, M_i is a m.f.t. subshift.

We now define \bar{M} as

$$\bar{M} := \bigcap_{i \geq 1} M_i.$$

For each $1 \leq j \leq k$, by (9.87), (9.81) and the selection of δ_i , we see

$$\rho(\nu_i^j, \nu_{i-1}^j) < 3^{-i-1} \varepsilon, i \in \mathbb{N}. \quad (9.91)$$

So $\{\nu_i^j\}_{i \in \mathbb{N}}$ constitutes a Cauchy sequence. Let ν^j denotes its limit.

Lemma 9.9. We have the following facts

- (1) \overline{M} is minimal.
- (2) Each ν^j is σ -invariant and supported on \overline{M} .
- (3) $h_{\nu^j} > h_{\mu^j} - \eta, j = 1, \dots, k$.
- (4)

$$\rho(\nu^j, \mu^j) < \varepsilon/2, j = 1, \dots, k. \quad (9.92)$$

Proof. (1) Observe that \overline{M} can be characterized as $Bl_{t_i}(\overline{M}) = Bl_{t_i}(M_i)$. Indeed, $Bl_{t_i}(\overline{M}) \subseteq Bl_{t_i}(M_i)$ is obvious since $\overline{M} \subseteq M_i$. On the other hand, for every $P \in Bl_{t_i}(M_i)$, one has $P \prec M_{i+1}$. Inductively, one gets that $P \prec M_j$ for any $j \geq i$. Consequently, one has $P \prec \overline{M} = \bigcap_{j \geq i} M_j$ since $\{M_i\}_{i \in \mathbb{N}}$ is a decreasing sequence. Therefore, one has $Bl_{t_i}(\overline{M}) \supset Bl_{t_i}(M_i)$. Then minimality of \overline{M} follows from (9.86) and lemma 9.2.

(2) Note that the space of σ -invariant measures on M are closed, so each ν^j is σ -invariant. Moreover, for each $1 \leq j \leq k$ and $i \in \mathbb{N}$, $1 = \limsup_{i \rightarrow \infty} \nu_i^j(M_i) \leq \nu^j(M_i)$ [75]. So $\mu^j(M_i) = 1$ for each i , which implies that $\nu^j(\overline{M}) = \nu^j(\bigcap_{i \in \mathbb{N}} M_i) = 1$. However, \overline{M} is minimal. Thus $S_{\nu^j} = \overline{M}$.

(3) Since M is expansive, the entropy map is upper semi-continuous [75]. So one has

$$h_{\nu^j} \geq \limsup_{i \rightarrow \infty} h_{\nu_i^j} > h_{\mu^j} - \eta.$$

(4) Indeed, (9.82) and (9.91) indicate that

$$\rho(\nu^j, \mu^j) \leq \sum_{i=1}^{\infty} 3^{-i} \varepsilon = \varepsilon/2.$$

□

Let $K := \text{cov}\{\nu^j\}_{j=1}^k$.

Lemma 9.10. $\mathfrak{M}_\sigma(\overline{M}) \subseteq K$.

Proof. Since $\mathfrak{M}_\sigma(\overline{M})$ and K are convex, we only need to show that for any ergodic measure $\nu \in \mathfrak{M}_\sigma(\overline{M})$, $\nu \in K$. Let $\kappa > 0$. There exist $t \in \mathbb{N}$ and $\eta > 0$ such that for $\alpha, \alpha' \in \mathfrak{M}_\sigma(\overline{M})$:

$$\forall P \in Bl_t(M) : |\alpha(P) - \alpha'(P)| < 4\eta \Rightarrow \rho(\alpha, \alpha') < \kappa. \quad (9.93)$$

Let i_0 is so large that $8 \cdot 2^{-i_0+1} < \eta$ and $2^{i_0-1} \geq t$. Then for all $i \geq i_0, r \geq \overline{r}_i$, one has

$$t \leq 2^{i_0-1} \leq 2^{i-1} < t_{i-1} \text{ and } 8\varepsilon_{i-1} < 8 \cdot 2^{-i+1} \leq 8 \cdot 2^{-i_0+1} < \eta.$$

Therefore, for any $P \in Bl_t(M), Q \in Bl_r(M_i^j)$, one has

$$|\mu_Q(P) - \nu_{i-1}^j(P)| < \delta_{i-1} < \eta \text{ and thus } |\nu_i^j(P) - \nu_{i-1}^j(P)| < \delta_{i-1} < \min\{2^{-i+1}, \eta\}.$$

Consequently,

$$\begin{aligned} & |\mu_Q(P) - \nu^j(P)| \\ & \leq |\mu_Q(P) - \nu_{i-1}^j(P)| + \sum_{k=i}^{\infty} |\nu_{k-1}^j(P) - \nu_k^j(P)| \\ & \leq \eta + \sum_{k=i}^{\infty} 2^{-k+1} = \eta + 2 \cdot 2^{-i+1} < \eta + \eta = 2\eta. \end{aligned}$$

Now choose a generic point $w \in \overline{M}$ of ν . Then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $P \in Bl_t(M)$

$$|\mu_{w_{\langle 0, n-1 \rangle}}(P) - \nu(P)| = \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_P(\sigma^i(w)) - \nu(P) \right| < \eta. \quad (9.94)$$

Choose an $i \geq i_0$ such that $\bar{r}_i \geq n_0$. Then choose an $n \in \mathbb{N}$ such that

$$\bar{r}_i < n < \varepsilon_{i-1} \cdot r_i.$$

This is possible by the selection of r_i . Since all the transition blocks U_i^{st} are recognizable by the block C_i , the sequence w can be split up uniquely into the transition blocks and into blocks belonging to the subshifts M_i^j , the latter with length at least $\frac{3}{4}r_i$, so we have the following two cases:

a) $w_{\langle 0, n \rangle}$ falls completely into a M_i^j -piece. Then $w_{\langle 0, n-1 \rangle} \in Bl_n(M_i^j)$ with $n > \bar{r}_i$. So

$$|\mu_{w_{\langle 0, n-1 \rangle}}(P) - \nu^j(P)| < 2\eta.$$

This along with (9.94) indicates that

$$|\nu(P) - \nu^j(P)| < 3\eta \text{ for any } P \in Bl_t(M).$$

Hence $d(\nu, \nu^j) < \kappa$ by (9.93).

b) $w_{\langle 0, n-1 \rangle}$ overlaps one of the transition blocks U . Then at the right of U begins a M_i^j -piece. For the relative frequency $\mu_{w_{\langle 0, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}}$, only the part $w_{\langle n+L_i, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}$ which belongs to a M_i^j -block is essential. This is due to $(n+L_i) < \varepsilon_{i-1} \cdot r_i + \varepsilon_{i-1} \cdot r_i = 2\varepsilon_{i-1} \cdot r_i$. More exactly, for $P \in Bl_t(\overline{M})$,

$$\mu_{w_{\langle 0, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}}(P) = \frac{(n+L_i)\mu_{w_{\langle 0, n+L_i+r-1 \rangle}}(P) + (\lceil \frac{r_i}{2} \rceil - r + 1)\mu_{w_{\langle n+L_i, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}}(P)}{n+L_i + \lceil \frac{r_i}{2} \rceil - r + 1}.$$

This yields that

$$\left| \mu_{w_{\langle 0, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}}(P) - \mu_{w_{\langle n+L_i, n+L_i+\lceil \frac{r_i}{2} \rceil-1 \rangle}}(P) \right| < 2 \cdot \frac{n+L_i}{\lceil \frac{r_i}{2} \rceil} < 8\varepsilon_{i-1} < \eta. \quad (9.95)$$

Now we employ the argument in case a) and get that

$$\left| \mu_{w_{\langle n+L_i, n+L_i+\lceil \frac{r_i}{2} \rceil \rangle}}(P) - \nu^j(P) \right| < 2\eta. \quad (9.96)$$

In the light of (9.94), (9.95) and (9.96), one gets $|\nu(P) - \nu^j(P)| < 4\eta$ which implies again $d(\nu, \nu^j) < \kappa$ by (9.93).

Since $\kappa > 0$ can be chosen arbitrarily small, one sees that ν must coincide with some ν^j with $1 \leq j \leq k$. This proves that there are no other ergodic measures supported on \overline{M} than $\{\nu^j\}_{j=1}^k$. Since K is convex, we obtain $\mathfrak{M}_\sigma(\overline{M}) \subseteq K$. \square

Lemma 9.11. $\mathfrak{M}_\sigma(\overline{M}) = K$. In particular, \overline{M} supports exactly k ergodic measures $\{\nu^j\}_{j=1}^k$.

Proof. By (9.92), for any $1 \leq a \leq k$,

$$\text{dist}(\text{cov}\{\mu^b\}_{1 \leq b \leq k, b \neq a}, \text{cov}\{\nu^b\}_{1 \leq b \leq k, b \neq a}) < \varepsilon/2. \quad (9.97)$$

This along with (4.12) implies for any $1 \leq a \leq k$ that

$$\begin{aligned}
& \text{dist}(\nu^a, \text{cov}\{\nu^b\}_{1 \leq b \leq k, b \neq a}) \\
& \geq \text{dist}(\mu^a, \text{cov}\{\mu^b\}_{1 \leq b \leq k, b \neq a}) - \text{dist}(\nu^a, \mu^a) - \text{dist}(\text{cov}\{\mu^b\}_{1 \leq b \leq k, b \neq a}, \text{cov}\{\nu^b\}_{1 \leq b \leq k, b \neq a}) \\
& > \varepsilon - \varepsilon/2 - \varepsilon/2 = 0.
\end{aligned} \tag{9.98}$$

So if $\mathfrak{M}_\sigma(\overline{M}) \neq K$, then by lemma 9.10, there is some $1 \leq a \leq k$ such that

$$\nu^a \in \text{cov}\{\nu^b\}_{1 \leq b \leq k, b \neq a},$$

contradicting (9.98). \square

Therefore, \overline{M} has exactly k ergodic measures $\{\nu^j\}_{j=1}^k$ and the proof is completed. \square

Corollary 9.12. Consider a m.f.t.-subshift M and a continuous function φ on M . If $I_\varphi \neq \emptyset$, then for any $0 < h < h(M)$ and any $k \geq 2$, there exists a minimal subsystem $\overline{M} \subseteq M$ supporting exactly k ergodic measures $\{\nu^j\}_{j=1}^k$ such that

$$1 \quad \int \varphi d\nu^1 < \int \varphi d\nu^2 < \cdots < \int \varphi d\nu^k. \tag{9.99}$$

2 $h_{\nu^j} > h, j = 1, \dots, k$ and in particular, $h(\overline{M}) > h$.

Proof. By the variational principle, we select a σ -invariant measure μ such that $h_\mu > h$. Since $I_\varphi \neq \emptyset$, there is a σ -invariant measure ν such that $\int \varphi d\mu \neq \int \varphi d\nu$. Without loss of generality, we suppose $\int \varphi d\mu < \int \varphi d\nu$. Then we choose $0 < \theta_1 < \theta_2 < \cdots < \theta_k < 1$ close to 1 such that $\tilde{\mu}^j := \theta_j \mu + (1 - \theta_j) \nu, j = 1, \dots, k$ satisfy that

$$h_{\tilde{\mu}^j} \geq \theta_j h_\mu > h \text{ and } \int \varphi d\tilde{\mu}^1 < \int \varphi d\tilde{\mu}^2 < \cdots < \int \varphi d\tilde{\mu}^k.$$

Since M is transitive and topologically expanding, by the star-entropy-dense property, there are ergodic measures $\mu^j, j = 1, \dots, k$ satisfy that

$$h_{\mu^j} > h \text{ and } \int \varphi d\mu^1 < \int \varphi d\mu^2 < \cdots < \int \varphi d\mu^k.$$

Now we employ proposition 4.3 and obtain the required \overline{M} and $\{\nu^j\}_{j=1}^k$. \square

Acknowledgements. The authors are grateful to Prof. Yu Huang, Wenxiang Sun and Xiaoyi Wang for their numerous remarks and fruitful discussions. The research of X. Tian was supported by National Natural Science Foundation of China (grant no. 11671093).

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